Tubes in Hyperbolic 3-Manifolds

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Abstract

We prove a lower bound on the volume of a maximal precisely invariant tube in a hyperbolic 3-manifold. This lower bound depends on the radius of the tube and the angle between the directions of two copies of the tube at a point where the tube intersects itself tangentially. Using this, we produce a lower bound on the volume of any closed orientable hyperbolic 3-manifold.

1 Introduction

The main result of this paper is a new lower bound on the volume of closed orientable hyperbolic 3-manifolds. We also provide results concerning symmetry groups of hyperbolic 3-manifolds. These results have their origin in a new lower bound on the volume of a tube of radius r embedded in a hyperbolic 3-orbifold.

We shall start by discussing the basic terminology and concepts which will be required. *Hyperbolic 3-space*, $\mathbb{H}^3$ is the unique simply connected Riemannian 3-manifold with constant curvature equal to -1. There are various models for $\mathbb{H}^3$, but we will use the *upper half space* model in which $\mathbb{H}^3$ consists of points $(x_1, x_2, x_3) \in \mathbb{R}^3$ with $x_3 > 0$ with the metric given by $ds = \dfrac{|dx|}{x_3}$. Using this model, the set of *points at infinity*, $\partial\mathbb{H}^3$ is $\{\mathbb{R}^2 \times \{0\}\} \cup (0, 0, \infty)$, which can be viewed as the Riemann sphere.

Throughout this paper, we will take as an assumption that all isometries are orientation preserving. Any isometry of $\mathbb{H}^3$ can be extended to $\partial\mathbb{H}^3$. Depending on the number and location of fixed points, we classify a given nontrivial isometry into one of three classes.

i) A *parabolic* isometry fixes a single point of $\partial\mathbb{H}^3$.

ii) A *hyperbolic* isometry fixes no points of $\mathbb{H}^3$ but leaves invariant some line in $\mathbb{H}^3$ and thus its endpoints in $\partial\mathbb{H}^3$. This line is referred to as the *axis* of the isometry.

iii) An *elliptic* isometry has fixed points in $\mathbb{H}^3$. The existence of a single fixed point implies the existence of an entire line of fixed points. This line is again referred to as the axis.

A *Kleinian group* Γ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$ which is not virtually abelian. If $\mathbb{H}^3/\Gamma$ is compact, then Γ contains no parabolic elements. If Γ contains no elliptic elements, then Γ is torsion free and $\mathbb{H}^3/\Gamma$ is a manifold. Our interest will be the volume of $\mathbb{H}^3/\Gamma$. (Note: we will use 3-manifold to refer to an object which is known to be a manifold and 3-fold to refer to an object which may be either an orbifold or a manifold.)

A subset $X \subset \mathbb{H}^3$ is *precisely invariant* under the action of a Kleinian group Γ if for any $\gamma \in \Gamma$, either $(\gamma X) \cap X = \emptyset$ or $\gamma X = X$. A nonparabolic element of Γ is *simple* if its axis is precisely invariant under the action of Γ. A precisely invariant tube about the axis of a simple element projects to an embedded tube about a geodesic in $\mathbb{H}^3/\Gamma$. Our result places a lower bound on the volume of a maximal embedded tube about a geodesic. The existence of such tubes is guaranteed. In particular, the shortest geodesic in a hyperbolic 3-fold always has an embedded tube about it.

Now that we have the necessary terminology, we provide a brief discussion of prior results about volumes of hyperbolic 3-folds.

For finite volume hyperbolic 3-folds, Mostow Rigidity [Mos73] shows that the volume is in fact a topological invariant. This allows one to talk about the set of volumes of hyperbolic 3-folds. It can be shown that this set is actually well-ordered [Thu78], and thus there is a smallest volume hyperbolic 3-fold. The identity of this 3-fold is not yet known. However, much progress has been made, particularly in the noncompact case [Ada87][Mey86]. For example, the smallest noncompact hyperbolic 3-manifold is known to be the Gieseking manifold [Ada87]. Its volume is $1.01\ldots$.

The compact case has proven to be more difficult. In general, the best results, thus far, provide lower bounds for the volume of a hyperbolic 3-fold given certain topological or geometric restrictions. The smallest known closed hyperbolic 3-manifold is an example due to Weeks, which we shall refer to as the Weeks manifold. It has volume $0.9427\ldots$. Until recently, the best lower bounds on the volume of an arbitrary closed hyperbolic 3-manifold were orders of magnitude away from the Weeks manifold [Mey86][GM91]. Given some information about the manifold, these lower bounds can be vastly improved [Ago][CS97][CS93]. For example, Culler, Hersonsky, and Shalen have shown that the smallest closed hyperbolic 3-manifold has Betti number at most 2 [CHS98].

Recently, Gehring and Martin [GM98] provided a lower bound on the volume of any hyperbolic 3-fold containing an embedded tube about one of its geodesics. This lower bound depends on the radius of the tube. They achieved this result by considering a packing of balls around the tube. They refer to the projection of one of these balls onto the boundary of the tube as the shadow. The packing of balls around the tube leads to a packing of the shadow in the boundary of the tube, which is a Euclidean cylinder. By locating an ellipse inside the shadow, they apply standard ellipse packing arguments to determine an upper bound on the density of this packing. This leads to a lower bound on the volume of the tube and hence a lower bound on the volume of the manifold.

Of course, in order to apply such a result, one would need to be able to produce a lower bound on the radius of an embedded tube. Fortunately, there are such lower bounds. Gehring, Maclachlan, Martin, and Reid provide one such set of lower bounds [GMMR97]. Their lower bounds require various assumptions about the nature of Γ. In addition, Gabai, Meyerhoff, and Thurston [GMT] show that practically all closed orientable hyperbolic 3-manifolds contain an embedded tube of radius at least $\frac{\log 3}{2}$. Combined with Gehring and Martin's result about tube volume, this establishes that any closed orientable hyperbolic 3-manifold has volume at least 0.1666.

Borrowing heavily from the techniques used by Gehring and Martin, we provide an improved lower bound on the volume of a tube. We do this by introducing an extra piece of information. If the tube is maximal, then it will intersect itself on the boundary. In addition to the radius of the tube, we consider the angle between the directions of the two sides of the tube at such an intersection point. Lifting to the universal cover, we then consider a packing of tubes about this central tube. Again, looking at the projection onto the boundary of the tube, we obtain a packing on the cylinder. By determining a lower bound on the area of a fundamental domain for this packing we determine a lower bound of the volume of $\mathbb{H}^3/\Gamma$.

Specifically, we prove:

Theorem 3.9: *If a closed hyperbolic 3-manifold M contains a maximal precisely invariant tube of radius $r \geq 0.42$ which meets itself at an angle θ, then the volume of M is at least* $V(r,\theta) = \sinh^2 r \int\limits_{z\in\Omega} \frac{dA}{|z^2 + \sinh^2(2r+i\theta)|}$ *where Ω is the ellipse bounded by* $\cosh(r+it)$ *for* $t \in [0, 2\pi]$.

There are similar versions of this theorem dealing with the case of orbifolds. Depending on the amount of 2-torsion, the lower bound is cut by a factor of 2 or 4. Regardless, this lower bound is somewhat difficult to compute. We use a Taylor series approximation to place a lower bound on the volume of any manifold.

Corollary 4.11: *Any closed orientable hyperbolic 3-manifold has volume at least* 0.276666*.*

We also provide lower bounds on the volumes of closed hyperbolic 3-manifolds which have symmetries of large order. In particular, we show that

Corollary 5.4: *The order of the symmetry group of the smallest volume hyperbolic 3-manifold is of the form* $2^m 3^n$ *for some* $m, n \geq 0$*.*

2 Lines in $\mathbb{H}^3$

We will need to consider the case of two tubes in $\mathbb{H}^3$ which are tangent to one another. To do this, we start by developing several basic properties of lines in $\mathbb{H}^3$, for which we use the upper half-space model. These properties are then applied to the axes of the two tubes.

Proposition 2.1. *The hyperbolic distance d between a line with endpoints* 0 *and* ∞ *and a line with endpoints λ and $\frac{1}{\lambda}$ is given by* $\cosh d = \frac{1+|\lambda|^2}{|1-\lambda^2|}$.

Proof. Let l_1 denote the line with endpoints 0 and ∞ and let l_2 be the line with endpoints λ and $\frac{1}{\lambda}$. The möbius transformation $z \to \frac{1}{z}$ will preserve both l_1 and l_2 so it must also preserve their common perpendicular. This common perpendicular is on a sphere centered at the origin. Since the only such sphere preserved by this möbius transformation is the unit sphere, the common perpendicular must lie on the unit sphere. Thus the points where the common perpendicular intersect l_1 and l_2 must be where l_1 and l_2 intersect the unit sphere. l_1 intersects the unit sphere at $(0,1)$. We can parametrize l_2 as

$$\left(\frac{1}{2}(\lambda+\frac{1}{\lambda})+\frac{1}{2}(\lambda-\frac{1}{\lambda})\cos t, \frac{1}{2}\left|\lambda-\frac{1}{\lambda}\right|\sin t\right) \text{ for } t\in(0,2\pi).$$

So we need to solve

$$\frac{1}{4}|\lambda+\frac{1}{\lambda}|^2+\frac{1}{4}|\lambda-\frac{1}{\lambda}|^2+\frac{1}{4}((\lambda+\frac{1}{\lambda})(\bar{\lambda}-\frac{1}{\bar{\lambda}})+(\bar{\lambda}+\frac{1}{\bar{\lambda}})(\lambda-\frac{1}{\lambda}))\cos t=1.$$

This gives $\cos t=\dfrac{1-|\lambda|^2}{1+|\lambda|^2}$ and $\sin t=\dfrac{2|\lambda|}{1+|\lambda|^2}$. Representing a point P in spherical coordinates (ρ,θ,ϕ), the hyperbolic distance d from P to l_1 is given by $\cosh d=\sec\phi$. For this particular point, $\cosh d=\sec\phi=\dfrac{1+|\lambda|^2}{|1-\lambda^2|}$. □

Definition 2.2. *Given two skew lines l_1 and l_2, we may form the plane Π which is normal to l_1 and contains the common perpendicular to the lines. Let P denote the point at which l_2 and Π intersect. We define the angle between the directions of the two lines to be the angle between l_2 and the normal to Π at P.*

We note that one could also define θ to be the imaginary part of the complex length of the isometry of $\mathbb{H}^3$ which carries l_1 to l_2 and leaves their common perpendicular invariant. The only distinction is that this alternative definition allows for values of θ which are not in $[0,\frac{\pi}{2}]$. Aside from this issue, the definitions are the same and are symmetric with respect to the order of l_1 and l_2.

Proposition 2.3. *The angle θ between the directions of a line l_1 with endpoints 0 and ∞ and a line l_2 with endpoints λ and $\frac{1}{\lambda}$ is given by $\cos\theta=\dfrac{|1-|\lambda|^2|}{|1-\lambda^2|}$.*

Proof. As in the proof of Proposition 2.1, l_2 is parametrized as

$$(\frac{1}{2}(\lambda+\frac{1}{\lambda})+\frac{1}{2}(\lambda-\frac{1}{\lambda})\cos t, \frac{1}{2}\left|\lambda-\frac{1}{\lambda}\right|\sin t).$$

The point determined by $\cos t = \dfrac{1-|\lambda|^2}{1+|\lambda|^2}$ is the base of the common perpendicular to the two lines. Let $\mathbf{v_1}$ be the tangent vector to l_2 at this point. Then

$$\begin{aligned}
\mathbf{v_1} &= (-\mathrm{Re}(\frac{1}{2}(\lambda - \frac{1}{\lambda})\sin t), -\mathrm{Im}(\frac{1}{2}(\lambda - \frac{1}{\lambda})\sin t), \frac{1}{2}\left|\lambda - \frac{1}{\lambda}\right|\cos t) \\
&= (\frac{|\lambda|}{1+|\lambda|^2}\mathrm{Re}(\frac{1}{\lambda} - \lambda), \frac{|\lambda|}{1+|\lambda|^2}\mathrm{Im}(\frac{1}{\lambda} - \lambda), \frac{1}{2}\left|\lambda - \frac{1}{\lambda}\right|\frac{1-|\lambda|^2}{1+|\lambda|^2}) \\
&= (\frac{1-|\lambda|^2}{|\lambda|(1+|\lambda|^2)}\mathrm{Re}(\lambda), -\frac{1}{|\lambda|}\mathrm{Im}(\lambda), \frac{1}{2}\frac{|1-\lambda^2|}{|\lambda|}\frac{1-|\lambda|^2}{1+|\lambda|^2}).
\end{aligned}$$

It is easy to see that the length of $\mathbf{v_1}$ is $\dfrac{1}{2}\left|\lambda - \dfrac{1}{\lambda}\right|$.

The unit sphere centered at the origin is the hyperbolic plane which is normal to l_1 and contains the common perpendicular of l_1 and l_2. Let $\mathbf{v_2}$ be the normal direction to this plane at the point P. Then

$$\mathbf{v_2} = (\sin\phi, 0, \cos\phi) = (\frac{\lambda + \bar{\lambda}}{1+|\lambda^2|}, 0, \frac{|1-\lambda^2|}{1+|\lambda^2|}).$$

We would like to compute $\cos\theta$ via the dot product of $\mathbf{v_1}$ and $\mathbf{v_2}$, but we must be careful about the sign of $\cos\theta$. Since $\theta \in [0, \frac{\pi}{2}]$, we have that $\cos\theta \geq 0$. However, $\mathbf{v_1} \cdot \mathbf{v_2}$ might be negative, depending on the orientations we have chosen for l_2. Hence

$$\begin{aligned}
\cos\theta &= \frac{1}{|\mathbf{v_1}|}|\mathbf{v_1} \cdot \mathbf{v_2}| \\
&= \frac{|1-|\lambda^2||}{|1-\lambda^2|(1+|\lambda|^2)^2}((\lambda + \bar{\lambda})^2 + |1-\lambda^2|^2) = \frac{|1-|\lambda^2||}{|1-\lambda^2|}.
\end{aligned}$$

□

Given two lines in the upper half-space model of $\mathbb{H}^3$, it is possible to perform an isometry of $\mathbb{H}^3$ such that the endpoints of one line are 0 and ∞ and the endpoints of the other line are λ and $\frac{1}{\lambda}$. Further, by performing reflections, if necessary, we may suppose that λ lies in the first quadrant portion of the unit disk. We make this choice in order to avoid ambiguities in the signs of various quantities. Also, we will eventually be dealing with a specific tube about l_2. In making this choice, we force the points of this tube to have positive x-coordinates.

So it is possible to associate a number $\lambda(r, \theta) \in \mathbb{C}$ with any two lines which are a distance $2r$ from one another and whose directions are separated by an angle θ.

Proposition 2.4. $\lambda(r, \theta) = \dfrac{\sinh 2r + i \sin\theta}{\cosh 2r + \cos\theta}$.

Proof. Recalling Propositions 2.1 and 2.3, $\cosh 2r = \dfrac{1 + |\lambda^2|}{|1 - \lambda^2|}$ and $\cos\theta = \dfrac{1 - |\lambda^2|}{|1 - \lambda^2|}$. Thus, $\cosh 2r + \cos\theta = \dfrac{2}{|1 - \lambda^2|}$. Further, $\sinh 2r = \dfrac{2\mathrm{Re}(\lambda)}{|1 - \lambda^2|}$ and $\sin\theta = \dfrac{2\mathrm{Im}(\lambda)}{|1 - \lambda^2|}$. By rearranging, it follows that

$$\mathrm{Re}(\lambda) = \frac{|1 - \lambda^2|}{2} \cdot \sinh 2r = \frac{\sinh 2r}{\cosh 2r + \cos\theta}$$

and $\mathrm{Im}(\lambda) = \dfrac{|1 - \lambda^2|}{2} \sin\theta = \dfrac{\sin\theta}{\cosh 2r + \cos\theta}$. Hence $\lambda = \dfrac{\sinh 2r + i \sin\theta}{\cosh 2r + \cos\theta}$. □

Definition 2.5. *Given points $u, v, w, z \in \mathbb{C}$ their cross ratio is*

$$R(u, v, w, z) = \frac{(w - u)(z - v)}{(w - v)(z - u)}.$$

It is well known that the cross ratio is invariant under fractional linear transformations.

Proposition 2.6. *Given a hyperbolic line with endpoints u and v and a hyperbolic line with endpoints w and z, the distance d between them is determined by* $\cosh d = \dfrac{|R| + 1}{|1 - R|}$ *where $R = R(u, v, w, z)$.*

Proof. By performing a fractional linear transformation, we may assume that $u = 0, v = \infty, w = \lambda$, and $z = \frac{1}{\lambda}$. Then $R(u, v, w, z) = \lambda^2$. By Proposition 2.1, $\cosh d = \dfrac{1 + |\lambda|^2}{|1 - \lambda^2|} = \dfrac{|R| + 1}{|1 - R|}$. □

3 The Shadow

Let Γ be a Kleinian group. We will be interested in the volume of $\mathbb{H}^3/\Gamma$. In the event that Γ is torsion free, i.e. $\mathbb{H}^3/\Gamma$ is a manifold, we will be able to develop stronger results, but for now, we do not include this assumption.

Our interest will be a maximal tube about a geodesic in $\mathbb{H}^3/\Gamma$. A priori, an arbitrary geodesic may intersect itself, but in a closed hyperbolic 3-fold there is at least one geodesic, the shortest one, which will not intersect itself and will in fact have a maximal tube of nonzero radius about it. Let T be the lift of such a tube to $\mathbb{H}^3$. We may take l_1, the axis of T, to be the line with endpoints 0 and ∞. Since the tube is maximal, it must intersect itself tangentially. Hence T must intersect some Γ translate of itself. Let $\gamma \in \Gamma$ be such a translation and let l_2 be the axis of γT.

Definition 3.1. *Let W be the union of all line segments perpendicular to l_1 and having one endpoint on l_1 and the other in some set X disjoint from T. The shadow of X on T is $W \cap \partial T$. In particular, we define S to be the shadow of γT on T.*

One simple property which can be seen from the definition is that there is some plane passing through l_1 such that γT, and hence S lie on one side of the plane. This will be important as it shows that S doesn't "wrap around" ∂T and that we can thus lift S isometrically to the universal cover of ∂T, the Euclidean plane.

Proposition 3.2. *If a line l perpendicular to l_1 passes through S, then the distance between l and l_2 is no more than r. Further, l passes through the boundary of S if and only if the distance between l and l_2 is exactly r.*

Proof. If l passes through S, then l passes through γT and hence contains some point which is within r of l_2. If this point is an interior point of γT, then there is an open neighborhood of this point which is contained within γT. Then the shadow of this neighborhood contains a neighborhood of $l \cap S$. Hence if l passes through ∂S, it passes through no interior point of γT. Thus it must be tangent to the boundary of γT. Likewise, if a line is perpendicular to l_1 and tangent to the boundary of γT, then it passes through ∂S. $\square$

The set S will lie in ∂T which has a natural Euclidean structure as a cylinder. We take (x, y) as coordinates where x measures distance along the direction of the axis and y measures distance in the perpendicular direction.

In order to determine the shadow of γT on T, we will need to consider the angle θ between the directions l_1 and l_2.

Proposition 3.3. *The region S is congruent to the region in the xy-plane which is bounded by the curve* $\frac{x}{\cosh r} + i\frac{y}{\sinh r} = \sinh^{-1}\frac{\cosh(r+it)}{\sinh(2r+i\theta)}$ *for* $t \in [0, 2\pi]$.

Proof. We may choose l_1 to be the line in upper half space whose endpoints are 0 and ∞ and l_2 to be the line with endpoints $\lambda(r,\theta)$ and $\frac{1}{\lambda(r,\theta)}$. Then a line l is perpendicular to l_1 if and only if its endpoints are additive inverses. Let the endpoints be z and $-z$. With $R = R(z, -z, \lambda, \frac{1}{\lambda})$, it follows that l passes through ∂S if and only if $\cosh r = \frac{|R|+1}{|1-R|}$. Using a double angle formula and writing $|R|+1$ as a sum of two squares, we have

$$2\cosh^2\frac{r}{2} - 1 = \cosh r = \frac{1}{2}\frac{|1+\sqrt{R}|^2 + |1-\sqrt{R}|^2}{|1-(\sqrt{R})^2|},$$

and hence

$$\begin{aligned}\cosh^2\frac{r}{2} &= \frac{1}{4}\left(\left|\frac{1+\sqrt{R}}{1-\sqrt{R}}\right| + 2 + \left|\frac{1-\sqrt{R}}{1+\sqrt{R}}\right|\right) \\ &= \frac{1}{4}\left(\sqrt{\left|\frac{1+\sqrt{R}}{1-\sqrt{R}}\right|} + \sqrt{\left|\frac{1-\sqrt{R}}{1+\sqrt{R}}\right|}\right)^2.\end{aligned}$$

From this we see that $\sqrt{|\frac{1+\sqrt{R}}{1-\sqrt{R}}|} = e^{\pm\frac{r}{2}}$. Then there is some real number t such that $\frac{1+\sqrt{R}}{1-\sqrt{R}} = e^{\pm r+it}$. Solving for $\sqrt{R}$ gives $\sqrt{R} = \frac{e^{\pm r+it}-1}{e^{\pm r+it}+1} = \tanh\frac{\pm r+it}{2}$. We now consider the way in which R depends on z and λ. $R = \frac{(\lambda-z)(\frac{1}{\lambda}+z)}{(\lambda+z)(\frac{1}{\lambda}-z)}$. This is equivalent to $(z-\frac{1}{z}) = \frac{1-\lambda^2}{\lambda}\cdot\frac{R+1}{R-1}$. From this it is easy to see that

$$\begin{aligned}\sinh \log z &= \frac{1-\lambda^2}{2\lambda}\cdot\frac{R+1}{R-1} = \frac{1-\lambda^2}{2\lambda}\cdot\frac{\tanh^2\frac{\pm r+it}{2}+1}{\tanh^2\frac{\pm r+it}{2}-1}\\ &= -\frac{1-\lambda^2}{2\lambda}\cosh(\pm r+it).\end{aligned}$$

Now we simplify

$$\begin{aligned}&\frac{1-\lambda^2}{2\lambda} = \frac{(\cosh 2r+\cos\theta)^2-(\sinh 2r+i\sin\theta)^2}{2(\cosh 2r+\cos\theta)(\sinh 2r+i\sin\theta)}\\ &= \frac{(1+\cosh 2r\cos\theta)-i\sinh 2r\sin\theta}{(\cosh 2r+\cos\theta)(\sinh 2r+i\sin\theta)}\\ &= \frac{\sinh 2r(1+\cosh 2r\cos\theta-\sin^2\theta)-i\sin\theta(1+\cosh 2r\cos\theta+\sinh^2 2r)}{(\cosh 2r+\cos\theta)(\sinh^2 2r+\sin^2\theta)}\\ &= \frac{\sinh 2r\cos\theta-i\cosh 2r\sin\theta}{\sinh^2 2r+\sin^2\theta} = \frac{1}{\sinh(2r+i\theta)}.\end{aligned}$$

Returning to our computation of z, we have $\log z = -\sinh^{-1}\dfrac{\cosh(\pm r+it)}{\sinh(2r+i\theta)}$. The observant reader will notice that the previous statement requires a choice of branch cut for $\sinh^{-1}$. We take the standard choice with $\sinh^{-1}$ having domain $\mathbb{C}-\{iy : y\in\mathbb{R} \text{ and } |y|>1\}$ and range $\{z\in\mathbb{C} : |\mathrm{Im} z|\le\dfrac{\pi}{2}\}$. It is in making this choice that we discard an unwanted duplicate of the shadow which would appear on the opposite side of ∂T. This choice is the correct one because it then follows that $\log z$ is in the range of $\sinh^{-1}$ so z has positive real part, as do all points of γT.

We note that by changing the sign of t if necessary, we may write our equation as $\log z = -\sinh^{-1}\dfrac{\cosh(r+it)}{\sinh(2r+i\theta)}$ and further that since this curve is symmetric about the origin, we may write it as $\log z = \sinh^{-1}\dfrac{\cosh(r+it)}{\sinh(2r+i\theta)}$. In the natural Euclidean coordinates on ∂T this corresponds to

$$\frac{x}{\cosh r}+i\frac{y}{\sinh r} = \sinh^{-1}\frac{\cosh(r+it)}{\sinh(2r+i\theta)}.$$

$\square$

With this parametrization, we are capable of noticing a simple property of the shadow. As $\sinh^{-1}$ of an ellipse, the shadow has a unique center around which there is an order two rotational symmetry. This symmetry and the convexity of the shadow are the two properties that will concern us. As both of these properties are preserved under linear transformation, we choose to work with the region bounded by the curve $w(t) = \sinh^{-1} \dfrac{\cosh(r+it)}{\sinh(2r+i\theta)}$.

Proposition 3.4. *If $r \geq 0.42$, then S is a convex set.*

Proof. As indicated, we prove convexity of S by proving that the region bounded by $w(t) = \sinh^{-1} \dfrac{\cosh(r+it)}{\sinh(2r+i\theta)}$ is convex. It would suffice to show that $\mathrm{Im}\dfrac{w''}{w'} > 0$. First, let us compute w''. From the definition of w, we have that $\sinh w = \dfrac{\cosh(r+it)}{\sinh(2r+i\theta)}$, and hence that $w' \cosh w = \dfrac{i\sinh(r+it)}{\sinh(2r+i\theta)}$ and

$$w'' \cosh w + (w')^2 \sinh w = -\frac{\cosh(r+it)}{\sinh(2r+i\theta)} = -\sinh w.$$

Solving for w' and w'' gives $w' = \dfrac{i\sinh(r+it)}{\cosh w \sinh(2r+i\theta)}$ and $w'' = -(1+(w')^2)\tanh w$. Combining these two expressions results in

$$\begin{aligned}
\frac{w''}{w'} &= i\frac{\sinh w \sinh(2r+i\theta)}{\sinh(r+it)}(1+(w')^2) = i\coth(r+it)(1+(w')^2) \\
&= i\coth(r+it) + i\coth(r+it)\left(\frac{i\sinh(r+it)}{\cosh w\sinh(2r+i\theta)}\right)^2 \\
&= i\coth(r+it) - i\frac{\sinh(r+it)\cosh(r+it)}{\sinh^2(2r+i\theta)+\cosh^2(r+it)}.
\end{aligned}$$

Then

$$
\begin{aligned}
\operatorname{Im}\frac{w''}{w'} &= \operatorname{Im}\left[i\coth(r+it) - i\frac{\sinh(r+it)\cosh(r+it)}{\sinh^2(2r+i\theta)+\cosh^2(r+it)}\right] \\
&= \operatorname{Im}\left[i\frac{\cosh(r+it)\sinh(r-it)}{|\sinh(r+it)|^2} - i\frac{\sinh(2r+2it)}{\cosh(4r+2i\theta)+\cosh(2r+2it)}\right] \\
&= \frac{\sinh 2r}{2(\cosh^2 r - \cos^2 t)} \\
&\quad -\operatorname{Re}\left[\frac{\sinh(2r+2it)(\cosh(4r-2i\theta)+\cosh(2r-2it))}{|\cosh(4r+2i\theta)+\cosh(2r+2it)|^2}\right] \\
&= \frac{\sinh 2r}{\cosh 2r - \cos 2t} \\
&\quad -\operatorname{Re}\left[\frac{\sinh(2r+2it)\cosh(4r-2i\theta)+\frac{1}{2}(\sinh 4r+\sinh 4it)}{4|\cosh(3r+i(\theta+t))\cosh(r+i(\theta-t))|^2}\right] \\
&= \frac{\sinh 2r}{\cosh 2r - \cos 2t} \\
&\quad -\frac{\sinh 2r\cos 2t\cosh 4r\cos 2\theta + \cosh 2r\sin 2t\sinh 4r\sin 2\theta + \frac{1}{2}\sinh 4r}{4(\sinh^2 3r+\cos^2(\theta+t))(\sinh^2 r+\cos^2(\theta-t))} \\
&= \frac{\sinh 2r}{\cosh 2r - \cos 2t} \\
&\quad -\frac{\cos 2t\cosh 4r\cos 2\theta + 2\cosh^2 2r\sin 2t\sin 2\theta + \cosh 2r}{2(\sinh^2 3r+\cos^2(\theta+t))(\cosh 2r+\cos 2(\theta-t))}\sinh 2r.
\end{aligned}
$$

At this point, we look at one portion of the above expression.

$$
\begin{aligned}
&\frac{\cos 2t\cosh 4r\cos 2\theta + 2\cosh^2 2r\sin 2t\sin 2\theta + \cosh 2r}{\cosh 2r+\cos 2(\theta-t)} \\
&\quad = 2\cosh^2 2r - 1 + \frac{2\cosh 2r - 2\cosh^3 2r + \sin 2t\sin 2\theta}{\cosh 2r+\cos 2(\theta-t)} \\
&\quad \le 2\cosh^2 2r - 1 + \frac{2\cosh 2r - 2\cosh^3 2r + 1}{\cosh 2r + 1} \\
&\quad = \frac{2\cosh^2 2r + \cosh 2r}{\cosh 2r + 1}
\end{aligned}
$$

The inequality follows from the fact that the $2\cosh 2r - 2\cosh^3 2r \le -1$ and that we are thus dealing with a nonpositive expression divided by a positive one. To maximize this, we maximize both the numerator and the denominator.

Returning to the original computation,

$$\begin{aligned}
\operatorname{Im}\frac{w''}{w'} &\geq \frac{\sinh 2r}{\cosh 2r - \cos 2t} - \frac{2\cosh^2 2r + \cosh 2r}{2(\sinh^2 3r + \cos^2(\theta+t))(\cosh 2r + 1)} \sinh 2r \\
&\geq \frac{\sinh 2r}{\cosh 2r + 1} - \frac{2\cosh^2 2r + \cosh 2r}{2(\sinh^2 3r)(\cosh 2r + 1)} \sinh 2r \\
&= \frac{\sinh 2r}{2(\sinh^2 3r)(\cosh 2r + 1)} (2\sinh^2 3r - 2\cosh^2 2r - \cosh 2r).
\end{aligned}$$

It is easy to see that $(2\sinh^2 3r - 2\cosh^2 2r - \cosh 2r) > 0$ if $r \geq 0.42$.

□

We now employ a trick developed by Adams [Ada87] and modified in [GM98].

Proposition 3.5. *Suppose Γ does not contain a primitive order two elliptic element whose axis is tangent to ∂T at $T \cap \gamma T$. Then the shadow S' of $\gamma^{-1}T$ on T is a translate of S under the action of Stab(T), but is not a translate of S under the action of $\Gamma_T =$ Stab(T) $\cap\, \Gamma$.*

Proof. As usual, we take the axis of T to have endpoints 0 and ∞ and the axis of γT to have endpoints λ and $\frac{1}{\lambda}$. Then the endpoints of the axis of $\gamma^{-1}T$ are $\gamma^{-1}(0)$ and $\gamma^{-1}(\infty)$. By proper choice of λ, we may assume that $\gamma(0) = \lambda$ and $\gamma(\infty) = \frac{1}{\lambda}$. As cross ratios are preserved, it follows that

$$\begin{aligned}
\lambda^2 &= R(0, \infty, \lambda, \frac{1}{\lambda}) = R(0, \infty, \gamma(0), \gamma(\infty)) \\
&= R(\gamma^{-1}(0), \gamma^{-1}(\infty), 0, \infty) = \frac{\gamma^{-1}(0)}{\gamma^{-1}(\infty)}.
\end{aligned}$$

We now consider the fractional linear transformation $z \to \frac{\lambda}{\gamma^{-1}(0)} z$. This map fixes 0 and ∞, sends $\gamma^{-1}(0)$ to λ and sends $\gamma^{-1}(\infty)$ to $\frac{\lambda}{\gamma^{-1}(0)}\gamma^{-1}(\infty) = \frac{1}{\lambda}$. Hence this map is an element of Stab(T) which carries $\gamma^{-1}(T)$ to $\gamma(T)$.

All that remains to be seen is that S' is not a translate of S under the action of Γ_T. Suppose that there is some element $\gamma_2 \in \Gamma_T$ such that $\gamma_2(S) = S'$. So $\gamma_2(T \cap \gamma T) = T \cap \gamma^{-1}T$ as $T \cap \gamma T$ is the center of S and $T \cap \gamma^{-1}T$ is the center of S'. We know that $S' = \gamma_2(S)$ is the shadow of $\gamma_2(\gamma T)$ on T. We also know that S' is the shadow of $\gamma^{-1}(T)$ on T. Both $\gamma_2(\gamma(T))$ and $\gamma^{-1}(T)$ are tubes of radius r which are tangent to T at the same point. Thus we note that from the equation for the boundary of the shadow, it is evident that since $\gamma_2(\gamma(T))$ and $\gamma^{-1}(T)$ have the same shadow, they must meet T at the same angle. Then it must be the case that $\gamma_2(\gamma(T)) = \gamma^{-1}(T)$. Thus $(\gamma_2^{-1}\gamma^{-1})T = \gamma T$ and $(\gamma_2^{-1}\gamma^{-1})(\gamma T) = \gamma_2^{-1}T = T$ so $\gamma_2^{-1}\gamma^{-1}$ exchanges T and γT. Then $\gamma_2^{-1}\gamma^{-1}$ is a primitive order two elliptic with axis tangent to ∂T at $T \cap \gamma T$. $\square$

When we first chose γ, we took any element of Γ which carried T to the then unnamed tube which we are now calling γT. In the event that Γ does contain the order two elliptic mentioned in Proposition 3.5, we could have chosen that element to be γ. This would result in S' and S being identical. Thus, when the order two elliptic is present, we will assume that it is γ and that $S = S'$.

We wish to create a Γ_T tiling of ∂T with copies of S and S'. In order to do this, we need to know that there are no nontrivial elements of Γ_T which preserve S and that the shadows of different tubes on T do not intersect S.

Proposition 3.6. *If an element of Γ_T leaves S invariant, then it is either the identity or a primitive order two elliptic whose axis is perpendicular to ∂T at $T \cap \gamma T$.*

Proof. Any element of Γ_T which leaves S invariant must fix $T \cap \gamma T$, the center of S. There are only two isometries of $\mathbb{H}^3$ which stabilize a given tube and fix a given point on the boundary. One is the identity, the other the aforementioned elliptic. $\square$

There is, of course, a similar result for S'. Now we show that convexity of S is sufficient to establish that the shadows of different tubes do not overlap.

Proposition 3.7. *Let T_1 and T_2 be two tubes of radius r which intersect T in single points and which have disjoint interiors. If there is an element of $Stab(T)$ which carries T_1 to T_2, then the interiors of their shadows on T are disjoint if the shadows are convex.*

Proof. Let S_1 and S_2 be the interiors of the shadows of T_1 and T_2 on T. It is evident from the parametrization of the boundary of a shadow that S_1 and S_2 have order two rotational symmetries. Hence, they have well defined centers. Let p be the midpoint of the line segment joining their centers. The order two rotation about p will swap the centers of S_1 and S_2 and hence will swap S_1 and S_2. Since S_1 and S_2 are translates of one another, $S_1 \cap S_2$ is invariant under this rotation. Since $S_1 \cap S_2$ is the intersection of two convex sets, it too is convex. Hence, if S_1 and S_2 intersect, p must be in their intersection. So the infinite ray originating perpendicular to the axis of T and passing through p must also pass through both T_1 and T_2. By assumption, T_1 and T_2 intersect in at most a point of tangency. Hence the ray through p must pass through one of the tubes and then the other or lie on $\partial S_1 \cap \partial S_2$. We may assume that the ray passes through T_1 first. The rotation of order two about p extends to an action of $\mathbb{H}^3$ which swaps T_1 and T_2 while preserving the ray through p. Performing this action would lead to the consequence that the ray passes through T_2 first, which is a contradiction. ☐

Proposition 3.8. *If $r \geq 0.42$, then the interior of $S \cup S'$ is precisely invariant under the action of Γ_T. A fundamental domain for the Γ_T action has area at least $C \cdot$ Area(S) where C is*

i) 2 *if Γ has no primitive order two elliptics whose axes intersect ∂T*

ii) 1 *if Γ contains no Klein 4-group stabilizing a point of ∂T*

iii) $\dfrac{1}{2}$ *otherwise.*

Proof. If $r \geq 0.42$ then S and S' are convex. Thus, Proposition 3.7 implies that the various Γ_T translates of S and S' either have disjoint interiors or are identical. Further, no element of Γ_T carries S to S' unless $S = S'$. This confirms that $S \cup S'$ is precisely invariant. We now need to know the answers to two questions: What is Area$(S \cup S')$? What is the number n of elements of Γ_T that carry S to either S or S'? The fundamental domain for the Γ_T action must have area at least $\dfrac{1}{n}$Area$(S \cup S')$.

Again, by Proposition 3.7, either S and S' have disjoint interiors or they are identical. If they are identical, then by Proposition 3.5, there is a primitive order two elliptic element whose axis is tangent to ∂T at $T \cap \gamma T$. By Proposition 3.6, if there are nontrivial elements of Γ_T which stabilize S, then

Γ_T contains a primitive order two elliptic whose axis is perpendicular to ∂T at $T \cap \gamma T$. By Proposition 3.5, there are no elements of Γ_T which carry S to S' unless Γ_T contains the indicated elliptic element, in which case, $S = S'$. Thus, we see that n is either 1 or 2, depending on whether Γ_T contains a specific type or order two element.

Putting all of this together, if Γ_T contains no primitive order two elliptics whose axes intersect ∂T, then the fundamental domain has area at least $2\text{Area}(S)$. If Γ_T contains primitive order two elliptics whose axes pass through $T \cap \gamma T$, either tangentially or perpendicularly, but not both, then the fundamental domain has area at least $\text{Area}(S)$. If Γ_T contains both types of order two elliptics, then the fundamental domain has area at least $\frac{1}{2}\text{Area}(S)$. Finally, we note that two order two elliptics with perpendicular intersecting axes generate a Klein 4-group which stabilizes the intersection point. $\square$

The previous result establishes a lower bound on the area of a fundamental domain for a tiling of ∂T. However, the action which leads to this tiling extends to an action on the interior of T. We now place a lower bound on the volume of a fundamental domain for this action.

Theorem 3.9. *If Γ is a Kleinian group and $\mathbb{H}^3/\Gamma$ contains a maximal tube of radius $r \geq 0.42$ which meets itself at an angle θ, then the volume of $\mathbb{H}^3/\Gamma$ is at least $\frac{C}{2}$ times as large as $V(r,\theta) = \sinh^2 r \int\limits_{z\in\Omega} \frac{dA}{|z^2 + \sinh^2(2r+i\theta)|}$ where Ω is the ellipse bounded by $\cosh(r+it)$ for $t \in [0, 2\pi]$ and C is as in the previous proposition.*

Proof. By a result in [GM98], the volume of the tube in $\mathbb{H}^3/\Gamma$ is equal to $\frac{1}{2}\tanh r$ times the area of the boundary of the tube. The area of the boundary of the tube is equal to the area of a fundamental domain for the action of Γ_T on ∂T which is greater than $C\text{Area}(S)$. Since a linear map of determinant $\cosh r \sinh r$ takes the region bounded by $\sinh^{-1} \frac{\cosh(r+it)}{\sinh(2r+i\theta)}$ to S, it suffices to compute the area of this region. This region however, is the image under $z \to \sinh^{-1} \frac{z}{\sinh(2r+i\theta)}$ of the ellipse Ω which is inside the curve $\cosh(r+it)$.

So the volume is at least

$$\begin{aligned}
&(\frac{1}{2}\tanh r)(C\cosh r\sinh r\int\limits_{z\in\Omega}\frac{dA}{|z^2+\sinh^2(2r+i\theta)|}) \\
=\;& \frac{C}{2}\sinh^2 r\int\limits_{z\in\Omega}\frac{dA}{|z^2+\sinh^2(2r+i\theta)|}.
\end{aligned}$$

□

4 Computation

Ideally, we would show that $V(r,\theta)$ is increasing in r, allowing us to remove the word maximal from Theorem 3.9. However, $V(r,\theta)$ is a complicated function. We instead approximate V with a different, simpler function. As a first step toward this, we find an equivalent single integral form for V.

Proposition 4.1.

$$V(r,\theta)=\frac{\sinh^2 r}{2}\int_0^{2\pi}\sinh^{-1}\left(\frac{\rho^2(\phi+\frac{\arg\alpha^2}{2})}{|\alpha|^2|\sin 2\phi|}+\frac{\cos 2\phi}{|\sin 2\phi|}\right)-\sinh^{-1}\frac{\cos 2\phi}{|\sin 2\phi|}\,d\phi$$

where $\alpha(r,\theta)=\sinh(2r+i\theta)$ *and* $\rho^2(\phi)=\dfrac{2\cosh^2 r\sinh^2 r}{\cosh 2r-\cos 2\phi}$.

Proof. We use polar coordinates (ρ,ϕ) writing $z=\rho e^{i\phi}$. Then the ellipse $\cosh(r+it)$ is $\dfrac{\rho^2\cos^2\phi}{\cosh^2 r}+\dfrac{\rho^2\sin^2\phi}{\sinh^2 r}=1$. Solving for ρ gives

$$\rho^2(\phi)=\frac{\cosh^2 r\sinh^2 r}{\sinh^2 r\cos^2\phi+\cosh^2 r\sin^2\phi}=\frac{\cosh^2 r\sinh^2 r}{\sinh^2 r+\sin^2\phi}=\frac{2\cosh^2 r\sinh^2 r}{\cosh 2r-\cos 2\phi}.$$

So the integral becomes

$$\frac{V(r,\theta)}{\sinh^2 r}=\int\limits_{z\in\Omega}\frac{dA}{|z^2+\alpha^2|}=\int_0^{2\pi}\int_0^{\rho(\phi)}\frac{\rho\,d\rho\,d\phi}{|\rho^2e^{2i\phi}+\alpha^2|}.$$

Making the substitution $u = \rho^2$ gives

$$
\begin{aligned}
&\frac{1}{2}\int_0^{2\pi}\int_0^{\rho^2(\phi)} \frac{du\,d\phi}{|ue^{2i\phi}+\alpha^2|} \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\rho^2(\phi)} \frac{du\,d\phi}{\sqrt{u^2+2u(\mathrm{Re}(\alpha^2)\cos 2\phi+\mathrm{Im}(\alpha^2)\sin 2\phi)+|\alpha|^4}} \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\rho^2(\phi)} \frac{du\,d\phi}{\sqrt{u^2+2u|\alpha|^2\cos(2\phi-\arg\alpha^2)+|\alpha|^4}} \\
&= \frac{1}{2}\int_0^{2\pi}\int_0^{\rho^2(\phi)} \frac{du\,d\phi}{\sqrt{(u+|\alpha|^2\cos(2\phi-\arg\alpha^2))^2+|\alpha|^4\sin^2(2\phi-\arg\alpha^2)}} \\
&= \frac{1}{2}\int_0^{2\pi}\left(\sinh^{-1}\left(\frac{\rho^2(\phi)}{|\alpha|^2|\sin(2\phi-\arg\alpha^2)|}+\frac{\cos(2\phi-\arg\alpha^2)}{|\sin(2\phi-\arg\alpha^2)|}\right)\right. \\
&\quad\left. -\sinh^{-1}\frac{\cos(2\phi-\arg\alpha^2)}{|\sin(2\phi-\arg\alpha^2)|}\right)d\phi \\
&= \frac{1}{2}\int_0^{2\pi}\sinh^{-1}\left(\frac{\rho^2(\phi+\frac{1}{2}\arg\alpha^2)}{|\alpha|^2|\sin 2\phi|}+\frac{\cos 2\phi}{|\sin 2\phi|}\right)-\sinh^{-1}\frac{\cos 2\phi}{|\sin 2\phi|}\,d\phi.
\end{aligned}
$$

□

We perform the approximation by taking the Taylor series for $\sinh^{-1}$ based at the point $\dfrac{\cos 2\phi}{|\sin 2\phi|}$. In order to get a good approximation we need to consider up through the third order term. We must further show that the later terms are not too large. We define $V_n(r,\theta)$ for $n > 0$ to be the contribution of the nth order term, i.e.

$$V_n(r,\theta) = \frac{\sinh^2 r}{2n!}\int_0^{2\pi}\left(\left(\frac{d}{dx}\right)^n \sinh^{-1} x\Big|_{x=\frac{\cos 2\phi}{|\sin 2\phi|}}\right)\left(\frac{\rho^2(\phi+\frac{1}{2}\arg\alpha^2)}{|\alpha|^2|\sin 2\phi|}\right)^n d\phi.$$

It should be noted that there is no constant term in the series expansion.

We define $\tilde{V}(r,\theta) = V_1(r,\theta) + V_2(r,\theta) + V_3(r,\theta)$.

Proposition 4.2.

$$\tilde{V}(r,\theta) = \pi\sinh^3 r\cosh r\left(\frac{1}{|\alpha|^2}-\frac{\cos\arg\alpha^2}{4|\alpha|^4}+\frac{2+\cosh 4r}{96|\alpha|^6}+\frac{3\cos 2\arg\alpha^2}{32|\alpha|^6}\right)$$

Proof.

$$\begin{aligned}
V_1(r,\theta) &= \frac{\sinh^2 r}{2}\int_0^{2\pi} |\sin 2\phi| \left(\frac{\rho^2(\phi+\frac{1}{2}\arg\alpha^2)}{|\alpha|^2|\sin 2\phi|}\right) d\phi \\
&= \frac{\sinh^2 r}{2|\alpha|^2}\int_0^{2\pi} \rho^2(\phi+\frac{1}{2}\arg\alpha^2)\, d\phi = \frac{\sinh^2 r}{2|\alpha|^2}\int_0^{2\pi} \rho^2(\phi)\, d\phi \\
&= \frac{\sinh^2 r}{2|\alpha|^2}\int_0^{2\pi} \frac{2\cosh^2 r\sinh^2 r}{\cosh 2r-\cos 2\phi}\, d\phi = \frac{\pi\sinh^2 r\sinh 2r}{2|\alpha|^2}
\end{aligned}$$

Likewise

$$\begin{aligned}
V_2(r,\theta) &= \frac{\sinh^2 r}{4}\int_0^{2\pi} \left(-|\sin 2\phi|^2\cos 2\phi\right)\left(\frac{\rho^2(\phi+\frac{1}{2}\arg\alpha^2)}{|\alpha|^2|\sin 2\phi|}\right)^2 d\phi \\
&= -\frac{\sinh^2 r}{4|\alpha|^4}\int_0^{2\pi} (\cos 2\phi)\rho^4(\phi+\frac{1}{2}\arg\alpha^2)\, d\phi \\
&= -\frac{\sinh^2 r}{4|\alpha|^4}\int_0^{2\pi} \cos(2\phi-\arg\alpha^2)\rho^4(\phi)\, d\phi \\
&= -\frac{\sinh^2 r}{4|\alpha|^4}\int_0^{2\pi} (\cos 2\phi)(\cos\arg\alpha^2)\rho^4(\phi)\, d\phi \\
&= -\frac{\sinh^2 r\cos\arg\alpha^2}{4|\alpha|^4}\int_0^{2\pi} \cos 2\phi\left(\frac{2\cosh^2 r\sinh^2 r}{\cosh 2r-\cos 2\phi}\right)^2 d\phi \\
&= -\frac{\pi\sinh^2 r\sinh 2r\cos\arg\alpha^2}{8|\alpha|^4}.
\end{aligned}$$

In a similar fashion, one can see that

$$V_3(r,\theta) = \frac{\pi\sinh^2 r\sinh 2r}{64|\alpha|^6}\left(\frac{2+\cosh 4r}{3}+3\cos 2\arg\alpha^2\right).$$

Hence we have that

$$\begin{aligned}
&\tilde{V}(r,\theta) = V_1(r,\theta)+V_2(r,\theta)+V_3(r,\theta) \\
&= \frac{\pi\sinh^2 r\sinh 2r}{2}\left(\frac{1}{|\alpha|^2}-\frac{\cos\arg\alpha^2}{4|\alpha|^4}+\frac{(2+\cosh 4r)}{96|\alpha|^6}+\frac{3\cos 2\arg\alpha^2}{32|\alpha|^6}\right).
\end{aligned}$$

□

In many of the results that follow it will be useful to divide the work into two cases depending on the size of $\arg\alpha^2$. Thus, we first develop some simple facts about $\arg\alpha^2$.

Proposition 4.3. *i)* $\arg\alpha^2 = \frac{\pi}{2}$ *if* $\tan\theta = \tanh 2r$.

ii) If $\arg\alpha^2 \in [0, \frac{\pi}{2}]$*, then* $|\alpha|^2 \geq \sinh^2 2r$.

iii) If $\arg\alpha^2 \in [\frac{\pi}{2}, \pi]$*, then* $|\alpha|^2 \geq \dfrac{2\sinh^2 2r\cosh^2 2r}{\cosh 4r}$.

iv) If $\arg\alpha^2 \in [\frac{\pi}{4}, \frac{\pi}{2}]$*, then* $|\alpha|^2 \geq \sinh^2 2r + \sin^2(\tan^{-1}\tanh 2r\tan\frac{\pi}{8})$.

Proof. Since $\arg\alpha^2 = 2\arg\alpha = 2\tan^{-1}(\coth 2r\tan\theta)$ the first result is obvious. It is also easy to see that $|\alpha|^2 = \sinh^2 2r + \sin^2\theta$ which is increasing in θ. As $\arg\alpha^2$ is also increasing in θ, the second result follows immediately by letting $\theta = 0$. For the third result, we may let $\theta = \tan^{-1}\tanh 2r$. Then

$$\begin{aligned} |\alpha|^2 &= \sinh^2 2r + \sin^2\theta = \sinh^2 2r + \frac{\sinh^2 2r}{\cosh 4r} \\ &= \sinh^2 2r\frac{\cosh 4r + 1}{\cosh 4r} = \frac{2\sinh^2 2r\cosh^2 2r}{\cosh 4r}. \end{aligned}$$

For the fourth result, we note that if $\arg\alpha^2 \geq \frac{\pi}{4}$, then $2\tan^{-1}\coth 2r\tan\theta \geq \frac{\pi}{4}$. □

The first three terms of the Taylor series are a good enough approximation for our purposes. However, it is difficult to prove that the fourth order remainder term is sufficiently small. Since we wish to determine a lower bound for V we place lower bounds on the fourth and fifth order terms and show that the sixth order remainder term is small.

Proposition 4.4. *For* $\arg\alpha^2 \in [\frac{\pi}{2}, \pi]$*, and* $\cosh 4r < 11$

$$V_4(r,\theta) \geq -\frac{\pi\sqrt{2}(11 - \cosh 4r)^{\frac{3}{2}}}{80\sinh^8 4r}\sinh^3 r\cosh r\cosh^4 4r.$$

In particular, for $r \geq \frac{\log 3}{2}$, $V_4(r,\theta) \geq -0.000571171$.

For $\arg\alpha^2 \in [\frac{\pi}{2}, \pi]$ *and* $\cosh 4r \geq 11$, $V_4(r,\theta) \geq 0$.

For $\arg\alpha^2 \in [0, \frac{\pi}{2}]$,

$$V_4(r,\theta) \geq -\frac{\pi(17+3\cosh 4r)}{256\sinh^8 2r}\sinh^3 r\cosh r.$$

In particular, for $r \geq \frac{\log 3}{2}$, $V_4(r,\theta) \geq -0.00837248$.

Proof. By computations similar to those in the proof of Proposition 4.2, we have that

$$V_4(r,\theta) = -\frac{\pi\sinh^3 r\cosh r}{512|\alpha|^8}\left(3(3+2\cosh 4r)\cos\arg\alpha^2 + 25\cos 3\arg\alpha^2\right).$$

We prove the result by placing an upper nonnegative bound on

$$\left(3(3+2\cosh 4r)\cos\arg\alpha^2 + 25\cos 3\arg\alpha^2\right)$$

and then applying the results of Proposition 4.3. This expression can be viewed as a function of $v = \arg\alpha^2$. We are then trying to maximize the function

$$(3(3+2\cosh 4r)\cos v + 25\cos 3v) \text{ for } v \in [0,\pi].$$

It is easy to see that the only potential maxima occur when $v = 0$ or when $v \in [\frac{\pi}{2}, \pi]$ and $\cos 2v = -\frac{14+\cosh 4r}{25}$. The maximum in the interval $v \in [0, \frac{\pi}{2}]$ must occur at 0 and the maximum in the interval $v \in [\frac{\pi}{2}, \pi]$ will occur when $\cos 2v = -\frac{14+\cosh 4r}{25}$ if there is such a point and at $v = \frac{\pi}{2}$ if there isn't. The values at these points are $34 + 6\cosh 4r$ and $\frac{2\sqrt{2}}{5}(11-\cosh 4r)^{\frac{3}{2}}$ and 0 respectively. □

Proposition 4.5.

$$V_5(r,\theta) \geq \min\left(0, \frac{3\pi\cosh r(-2213+244\cosh 4r+9\cosh 8r)\sinh^3 r}{250880|\alpha|^{10}}\right).$$

In particular, if $r \geq \frac{\log 3}{2}$, $\arg\alpha^2 \in [0, \frac{\pi}{2}]$ *then* $V_5(r,\theta) \geq -0.000346423$ *and if* $r \geq \frac{\log 3}{2}$, $\arg\alpha^2 \in [\frac{\pi}{2}, \pi]$ *then* $V_5(r,\theta) \geq -0.000128433$.

Proof. By computations similar to those in the proof of Proposition 4.2, we have that

$$\begin{aligned} V_5(r,\theta) &= \frac{\pi\cosh r\sinh^3 r}{40960|\alpha|^{10}}\big(162+144\cosh 4r+9\cosh 8r \\ &\quad +(400+300\cosh 4r)\cos 2\arg\alpha^2+1225\cos 4\arg\alpha^2\big). \end{aligned}$$

By similar reasoning to the proof of Proposition 4.4, it is sufficient to find the smallest nonpositive value on the interval $[\frac{\pi}{2},\pi]$ of the function

$$162+144\cosh 4r+9\cosh 8r+(400+300\cosh 4r)\cos 2v+1225\cos 4v.$$

It is easy to see that the only potential minima occur when $\cos 2v = -\dfrac{4+3\cosh 4r}{49}$ or, if there are no such points, when $v=\dfrac{\pi}{2}$. The value at $v=\dfrac{\pi}{2}$ is always positive so we consider only the other critical points, at which the value is

$$\frac{24}{49}(-2213+244\cosh 4r+9\cosh 8r).$$

This gives the desired result. $\square$

Proposition 4.6. *For* $r\geq \dfrac{\log 3}{2}$,

$$|V(r,\theta)-\sum_{n=1}^{5}V_n(r,\theta)| \leq \frac{\pi(38+24\cosh 4r+\cosh 8r)\sinh 4r\sinh^2 r}{1024(\sinh^2 2r+\sin^2\theta)^6}.$$

We define this upper bound to be $\epsilon(r,\theta)$.

Proof. We use the Cauchy form of the remainder which is given by

$$R_{n+1}=\frac{f^{(n+1)}(a+\zeta h)}{n!}h^{n+1}(1-\zeta)^n \text{ for some } \zeta\in[0,1].$$

By elementary calculus, one can show that $\dfrac{(1-\zeta)^5}{(1+(a+\zeta x)^2)^3}\leq\dfrac{1}{(1+a^2)^3}$ as long as $|x|\leq\dfrac{5}{6}\sqrt{a^2+1}$. For our purposes, $a=\dfrac{\cos 2\phi}{|\sin 2\phi|}$ and $x=\dfrac{\rho^2(\phi+\frac{1}{2}\arg\alpha^2)}{|\alpha|^2|\sin 2\phi|}$ and it is easy to see that $|x|\leq\dfrac{5}{6}\sqrt{a^2+1}$ if $r\geq\dfrac{\log 3}{2}$.

In the case of $\sinh^{-1}$, it is easy to see that $|\left(\frac{d}{da}\right)^6 \sinh^{-1} a| \leq \frac{120}{(1+a^2)^3}$ so

$$\begin{aligned}
|V(r,\theta) - \sum_{n=1}^{5} V_n(r,\theta)| &\leq \frac{\sinh^2 r}{2} \int_0^{2\pi} \frac{1}{(1+\frac{\cos^2 2\phi}{|\sin^2 2\phi|})^3} \left(\frac{\rho^2(\phi + \frac{1}{2}\arg\alpha^2)}{|\alpha|^2 |\sin 2\phi|}\right)^6 d\phi \\
&= \frac{\sinh^2 r}{2} \int_0^{2\pi} \left(\frac{\rho^2(\phi + \frac{1}{2}\arg\alpha^2)}{|\alpha|^2}\right)^6 d\phi \\
&= \frac{\sinh^2 r}{2|\alpha|^2} \int_0^{2\pi} \left(\rho^2(\phi)\right)^6 d\phi \\
&= \frac{\sinh^2 r}{2|\alpha|^{12}} \int_0^{2\pi} \left(\frac{2\cosh^2 r \sinh^2 r}{\cosh 2r - \cos 2\phi}\right)^6 d\phi.
\end{aligned}$$

Evaluating the integral gives the desired result. □

Now that we have lower bounds on the higher order terms, we need to determine the minimum value for $\tilde{V}(r,\theta)$.

Proposition 4.7. *$\tilde{V}(r,\theta) - \epsilon(r,\theta)$ is decreasing in θ for $\theta \in [0, \frac{\pi}{2}]$ and $r \geq \frac{\log 3}{2}$.*

Proof. We start by noting that $\frac{\partial |\alpha|^2}{\partial \theta} = 2\sin\theta\cos\theta$ and that

$$\begin{aligned}
\frac{\partial \arg\alpha^2}{\partial\theta} &= 2\frac{\partial \tan^{-1}\coth 2r \tan\theta}{\partial\theta} = \frac{\sinh 4r}{\sinh^2 2r \cos^2\theta + \cosh^2 2r \sin^2\theta} \\
&= \frac{\sinh 4r}{|\alpha|^2}.
\end{aligned}$$

We also note that

$$\sin\arg\alpha^2 = \frac{\mathrm{Im}\alpha^2}{|\alpha|^2} = \frac{2\sin\theta\cos\theta\cosh 2r \sinh 2r}{|\alpha|^2} = \frac{\sinh 4r}{2|\alpha|^2}\frac{\partial|\alpha|^2}{\partial\theta}.$$

Using the chain rule we then compute $\frac{\partial\tilde{V}}{\partial\theta} = \frac{\partial\tilde{V}}{\partial|\alpha|^2}\frac{\partial|\alpha|^2}{\partial\theta} + \frac{\partial\tilde{V}}{\partial\arg\alpha^2}\frac{\partial\arg\alpha^2}{\partial\theta}$,

$$\frac{\partial\tilde{V}}{\partial|\alpha|^2} = \pi\sinh^3 r\cosh r\left(-\frac{1}{|\alpha|^4} + \frac{\cos\arg\alpha^2}{2|\alpha|^6} - \frac{2+\cosh 4r}{32|\alpha|^8} - \frac{9\cos 2\arg\alpha^2}{32|\alpha|^8}\right)$$

and

$$
\begin{aligned}
\frac{\partial \tilde{V}}{\partial \arg\alpha^2} &= \pi \sinh^3 r \cosh r \left(\frac{\sin\arg\alpha^2}{4|\alpha|^4} - \frac{3\sin 2\arg\alpha^2}{16|\alpha|^6}\right) \\
&= \pi \sinh^3 r \cosh r \frac{\sin\arg\alpha^2}{4|\alpha|^4}\left(1 - \frac{3\cos\arg\alpha^2}{2|\alpha|^2}\right) \\
&= \pi \sinh^3 r \cosh r \frac{\sinh 4r}{8|\alpha|^6}\left(1 - \frac{3\cos\arg\alpha^2}{2|\alpha|^2}\right)\frac{\partial|\alpha^2|}{\partial\theta}.
\end{aligned}
$$

So we then have that

$$
\begin{aligned}
\frac{\partial \tilde{V}}{\partial \theta} &= \pi \sinh^3 r \cosh r \frac{\partial|\alpha|^2}{\partial\theta}\left(-\frac{1}{|\alpha|^4} + \frac{\cos\arg\alpha^2}{2|\alpha|^6} - \frac{2+\cosh 4r}{32|\alpha|^8}\right. \\
&\quad \left. - \frac{9\cos 2\arg\alpha^2}{32|\alpha|^8} + \frac{\sinh^2 4r}{8|\alpha|^8}\left(1 - \frac{3\cos\arg\alpha^2}{2|\alpha|^2}\right)\right) \\
&= -\frac{1}{|\alpha|^4}\frac{\partial|\alpha|^2}{\partial\theta}\pi \sinh^3 r \cosh r \left(1 - \frac{\cos\arg\alpha^2}{2|\alpha|^2} + \frac{2+\cosh 4r}{32|\alpha|^4}\right. \\
&\quad \left. + \frac{9\cos 2\arg\alpha^2}{32|\alpha|^4} - \frac{\sinh^2 4r}{8|\alpha|^4}\left(1 - \frac{3\cos\arg\alpha^2}{2|\alpha|^2}\right)\right).
\end{aligned}
$$

As $-\frac{1}{|\alpha|^4}\frac{\partial|\alpha|^2}{\partial\theta}\pi \sinh^3 r \cosh r$ is always negative, it will be sufficient to place a lower bound on the other term.

We now divide the proof into two cases. First, we assume that $0 \le \arg\alpha^2 \le \frac{\pi}{2}$. Since $\frac{\sinh^2 4r}{|\alpha|^4} \ge \frac{\sinh^2 4r \cosh^2 4r}{4\sinh^4 2r \cosh^4 2r} = 4\coth^2 2r \ge 4$, we may say that
$\frac{3\sinh^2 4r}{16|\alpha|^6}\cos\arg\alpha^2 \ge \frac{3}{4|\alpha|^2}\cos\arg\alpha^2$. In this case it is also useful to note that

$$
\begin{aligned}
\frac{\cos\arg\alpha^2}{|\alpha|^2} &= \frac{\mathrm{Re}\alpha^2}{|\alpha|^4} = \frac{\sinh^2 2r\cos^2\theta - \cosh^2 2r \sin^2\theta}{|\alpha|^4} \\
&= \frac{\sinh^2 2r - \cosh 4r \sin^2\theta}{|\alpha|^4} = \frac{2\sinh^2 2r\cosh^2 2r - |\alpha|^2\cosh 4r}{|\alpha|^4}.
\end{aligned}
$$

Hence

$$1-\frac{\cos\arg\alpha^2}{2|\alpha|^2}+\frac{2+\cosh 4r}{32|\alpha|^4}+\frac{9\cos 2\arg\alpha^2}{32|\alpha|^4}-\frac{\sinh^2 4r}{8|\alpha|^4}\left(1-\frac{3\cos\arg\alpha^2}{2|\alpha|^2}\right)$$
$$\geq \ 1-\frac{\cosh 4r}{4|\alpha|^2}+\frac{2+\cosh 4r}{32|\alpha|^4}+\frac{9\cos 2\arg\alpha^2}{32|\alpha|^4}.$$

At this point it becomes necessary to further divide into two subcases. If $\arg\alpha^2 \geq \frac{\pi}{4}$, then

$$\begin{aligned}
&1-\frac{\cosh 4r}{4|\alpha|^2}+\frac{2+\cosh 4r}{32|\alpha|^4}+\frac{9\cos 2\arg\alpha^2}{32|\alpha|^4}\\
&\geq \ 1-\frac{\cosh 4r}{4|\alpha|^2}+\frac{\cosh 4r}{32|\alpha|^4}-\frac{7}{32|\alpha|^4}\\
&= \ 1+\frac{1}{32|\alpha|^4}\left(\cosh 4r-8|\alpha|^2\cosh 4r-7\right)\\
&\geq \ 1+\frac{\cosh 4r-8(\sinh^2 2r+\sin^2(\tan^{-1}\tanh 2r\tan\frac{\pi}{8}))\cosh 4r-7}{32(\sinh^2 2r+\sin^2(\tan^{-1}\tanh 2r\tan\frac{\pi}{8}))^2}\\
&\geq \ 1+\frac{\cosh 4r-8(\sinh^2\log 3+\sin^2(\tan^{-1}\tanh\log 3\tan\frac{\pi}{8}))\cosh\log 9-7}{32(\sinh^2\log 3+\sin^2(\tan^{-1}\tanh\log 3\tan\frac{\pi}{8}))^2}\\
&\geq \ \frac{9380743-1253862\sqrt{2}}{20480000}.
\end{aligned}$$

In our second subcase, we take $\arg\alpha^2 \leq \frac{\pi}{4}$. Then

$$\begin{aligned}
&1-\frac{\cosh 4r}{4|\alpha|^2}+\frac{2+\cosh 4r}{32|\alpha|^4}+\frac{9\cos 2\arg\alpha^2}{32|\alpha|^4}\\
&\geq \ 1-\frac{\cosh 4r}{4|\alpha|^2}+\frac{2+\cosh 4r}{32|\alpha|^4}\\
&\geq \ 1+\frac{2+\cosh 4r-8|\alpha|^2\cosh 4r}{32|\alpha|^4}\\
&\geq \ 1+\frac{2+\cosh 4r-8\sinh^2 2r\cosh 4r}{32\sinh^4 2r}\\
&\geq \ 1+\frac{2+\cosh\log 9-8\sinh^2\log 3\cosh\log 9}{32\sinh^4\log 3}\\
&\geq \ \frac{3475}{8192}.
\end{aligned}$$

Now, we assume that $\frac{\pi}{2} \le \arg\alpha^2 \le \pi$. Since

$$\frac{\sinh^2 4r}{|\alpha|^4} \le \frac{\sinh^2 4r \cosh^2 4r}{4\sinh^4 2r \cosh^4 2r} = 4\coth^2 4r \le \frac{1681}{400}$$

we may say that $\frac{3\sinh^2 4r}{16|\alpha|^6}\cos\arg\alpha^2 \ge \frac{5043}{6400|\alpha|^2}\cos\arg\alpha^2$. Hence,

$$\begin{aligned}
&1 - \frac{\cos\arg\alpha^2}{2|\alpha|^2} + \frac{2+\cosh 4r}{32|\alpha|^4} + \frac{9\cos 2\arg\alpha^2}{32|\alpha|^4} - \frac{\sinh^2 4r}{8|\alpha|^4}\left(1 - \frac{3\cos\arg\alpha^2}{2|\alpha|^2}\right) \\
\ge\ & 1 + \frac{1843\cos\arg\alpha^2}{6400|\alpha|^2} + \frac{2+\cosh 4r}{32|\alpha|^4} + \frac{9\cos 2\arg\alpha^2}{32|\alpha|^4} - \frac{\sinh^2 4r}{8|\alpha|^4} \\
\ge\ & 1 - \frac{1843}{6400|\alpha|^2} + \frac{2+\cosh 4r}{32|\alpha|^4} - \frac{9}{32|\alpha|^4} - \frac{\sinh^2 4r}{8|\alpha|^4} \\
=\ & 1 - \frac{1843}{6400|\alpha|^2} + \frac{\cosh^2 2r}{16|\alpha|^4} - \frac{1}{4|\alpha|^4} - \frac{\sinh^2 4r}{8|\alpha|^4} \\
=\ & 1 - \frac{1843}{6400|\alpha|^2} - \frac{1}{4|\alpha|^4} + \frac{\cosh^2 2r(1 - 8\sinh^2 2r)}{16|\alpha|^4} \\
\ge\ & 1 - \frac{1843}{6400|\alpha|^2} - \frac{1}{4|\alpha|^4} + \frac{(1 - 8\sinh^2 2r)\cosh^2 4r}{64\sinh^4 2r\cosh^2 2r} \\
\ge\ & 1 - \frac{680067}{5120000} - \frac{136161}{2560000} - \frac{1}{2} = \frac{1607611}{5120000}.
\end{aligned}$$

Finally, we compute

$$\begin{aligned}
\frac{\partial\epsilon(r,\theta)}{\partial\theta} &= \frac{\partial\epsilon(r,\theta)}{\partial|\alpha|^2}\frac{\partial|\alpha|^2}{\partial\theta} \\
&= -\frac{6\pi(38 + 24\cosh 4r + \cosh 8r)\sinh 4r\sinh^2 r}{1024(\sinh^2 2r + \sin^2\theta)^7}\frac{\partial|\alpha|^2}{\partial\theta} \\
&= -\frac{1}{|\alpha|^4}\frac{\partial|\alpha|^2}{\partial\theta}\frac{6\pi(38 + 24\cosh 4r + \cosh 8r)\sinh 4r\sinh^2 r}{1024|\alpha|^{10}}.
\end{aligned}$$

When $0 \le \arg\alpha^2 \le \frac{\pi}{4}$, we have that this is at least

$$-\frac{1}{|\alpha|^4}\frac{\partial|\alpha|^2}{\partial\theta}\frac{6162075\pi}{67108864} \ge -\frac{1}{|\alpha|^4}\frac{\partial|\alpha|^2}{\partial\theta}\frac{2\pi}{9}\cdot\frac{3475}{8192}.$$

When $\frac{\pi}{4} \le \arg \alpha^2 \le \frac{\pi}{2}$, this is at least

$$\begin{aligned} & -\frac{1}{|\alpha|^4}\frac{\partial|\alpha|^2}{\partial\theta}\frac{246483(1152843658 + 533205009\sqrt{2})\pi}{6710886400000000} \\ & \qquad \ge \quad -\frac{1}{|\alpha|^4}\frac{\partial|\alpha|^2}{\partial\theta}\frac{2\pi}{9}\cdot\frac{9380743 - 1253862\sqrt{2}}{20480000}. \end{aligned}$$

When $\frac{\pi}{2} \le \arg \alpha^2 \le \pi$, this is at least

$$-\frac{1}{|\alpha|^4}\frac{\partial|\alpha|^2}{\partial\theta}\frac{28556583991083\pi}{838860800000000} \ge -\frac{1}{|\alpha|^4}\frac{\partial|\alpha|^2}{\partial\theta}\frac{2\pi}{9}\cdot\frac{1607611}{5120000}.$$

□

From this result it follows that for a given r, the value of $\tilde{V}(r,\theta)$ is minimized when $\theta = \frac{\pi}{2}$. This knowledge allows us to avoid proving that $\tilde{V}(r,\theta)$ is increasing in r for arbitrary values of θ. We may instead restrict to $\theta = \frac{\pi}{2}$.

Proposition 4.8. *$\tilde{V}(r, \frac{\pi}{2})$ is increasing in r.*

Proof.

$$\begin{aligned} \tilde{V}(r,\frac{\pi}{2}) &= \pi \sinh^3 r \cosh r \left(\frac{1}{\cosh^2 2r} + \frac{1}{4\cosh^4 2r} + \frac{11 + \cosh 4r}{96\cosh^6 2r}\right) \\ &= \frac{\pi}{4}(\cosh 2r - 1)\sinh 2r \left(\frac{1}{\cosh^2 2r} + \frac{13}{48\cosh^4 2r} + \frac{5}{48\cosh^6 2r}\right) \end{aligned}$$

which has derivative

$$
\begin{aligned}
\frac{d\tilde{V}}{dr} &= \frac{\pi}{2}\sinh^2 2r\left(\frac{1}{\cosh^2 2r}+\frac{13}{48\cosh^4 2r}+\frac{5}{48\cosh^6 2r}\right)\\
&\quad +\frac{\pi}{2}(\cosh 2r-1)\cosh 2r\left(\frac{1}{\cosh^2 2r}+\frac{13}{48\cosh^4 2r}+\frac{5}{48\cosh^6 2r}\right)\\
&\quad -\frac{\pi}{2}(\cosh 2r-1)\sinh^2 2r\left(\frac{2}{\cosh^3 2r}+\frac{13}{12\cosh^5 2r}+\frac{5}{8\cosh^7 2r}\right)\\
&= \pi\frac{2\cosh^2 2r-\cosh 2r-1}{2}\left(\frac{1}{\cosh^2 2r}+\frac{13}{48\cosh^4 2r}+\frac{5}{48\cosh^6 2r}\right)\\
&\quad -\frac{\pi}{2}(\cosh 2r-1)\sinh^2 2r\left(\frac{2}{\cosh^3 2r}+\frac{13}{12\cosh^5 2r}+\frac{5}{8\cosh^7 2r}\right)\\
&= \frac{\pi(\cosh 2r-1)}{2\cosh^2 2r}\left[(2\cosh 2r+1)\left(1+\frac{13}{48\cosh^2 2r}+\frac{5}{48\cosh^4 2r}\right)\right.\\
&\quad \left.-(\cosh^2 2r-1)\left(\frac{2}{\cosh 2r}+\frac{13}{12\cosh^3 2r}+\frac{5}{8\cosh^5 2r}\right)\right]\\
&= \frac{\pi(\cosh 2r-1)}{2\cosh^2 2r}\left(1+\frac{35}{24}\operatorname{sech} 2r+\frac{13}{48}\operatorname{sech}^2 2r\right.\\
&\quad \left.+\frac{2}{3}\operatorname{sech}^3 2r+\frac{5}{48}\operatorname{sech}^4 2r+\frac{5}{8}\operatorname{sech}^5 2r\right)\\
&\geq 0.
\end{aligned}
$$

□

The statements of Propositions 4.4 and 4.5 had two cases depending on the size of $\arg\alpha^2$. As a refinement of our lower bound for V, we also consider two cases for $\tilde{V}$.

Proposition 4.9. *$\tilde{V}(r,\tan^{-1}\tanh 2r)$ is increasing in r for $r \geq \dfrac{\log 3}{2}$.*

Proof.

$$\begin{aligned}
\tilde{V}(r,\tan^{-1}\tanh 2r) &= \frac{\pi\sinh^3 r\cosh r\cosh 4r}{2\sinh^2 2r\cosh^2 2r}\left(1+\frac{(\cosh 4r-7)\cosh^2 4r}{384\sinh^4 2r\cosh^4 2r}\right)\\
&= \frac{\pi\sinh^2 r\coth 4r}{2\cosh 2r}\left(1+\frac{(\cosh 4r-7)\cosh^2 4r}{24\sinh^4 4r}\right)\\
&= \frac{\pi}{96}\tanh r\coth 4r\left[\tanh 2r\left(24+\frac{(\cosh 4r-7)\cosh^2 4r}{\sinh^4 4r}\right)\right]
\end{aligned}$$

It is very easy to verify that $\tanh r\coth 4r$ is increasing so it remains to verify that the other factor is. Its derivative is

$$\begin{aligned}
&2\operatorname{sech}^2 2r\left(24+\frac{(\cosh 4r-7)\cosh^2 4r}{\sinh^4 4r}\right)\\
&\quad +4\frac{\tanh 2r}{\sinh^5 4r}\left(\sinh^2 4r(3\cosh^2 4r-14\cosh 4r)-4(\cosh 4r-7)\cosh^3 4r\right)\\
&= \frac{2\operatorname{sech}^2 2r}{\sinh^4 4r}\big(24\sinh^4 4r+(\cosh 4r-7)\cosh^2 4r\\
&\quad +\sinh^2 4r(3\cosh^2 4r-14\cosh 4r)-4(\cosh 4r-7)\cosh^3 4r\big)\\
&= \frac{2\operatorname{sech}^2 2r}{\sinh^4 4r}(23\cosh^4 4r+15\cosh^3 4r-58\cosh^2 4r+14\cosh 4r+24)\\
&\geq 0.
\end{aligned}$$

□

We are now capable of placing a lower bound on the volume of a tube of radius at least r. We illustrate this in the specific case of $r \geq \log 3/2$.

Proposition 4.10. *If $r \geq \log 3/2$, then $V(r,\theta) \geq 0.276666$.*

Proof. For $0 \leq \arg\alpha^2 \leq \frac{\pi}{2}$, we have that

$$\begin{aligned}
V(r,\theta) &\geq \tilde{V}(r,\theta)-\epsilon(r,\theta)-0.00837248-0.000346423\\
&\geq \tilde{V}(r,\tan^{-1}\tanh 2r)-\epsilon(r,\tan^{-1}\tanh 2r)-0.008718903\\
&\geq \tilde{V}(\frac{\log 3}{2},\tan^{-1}0.8)-\epsilon(\frac{\log 3}{2},0.8)-0.008718903\\
&\geq 0.320269-0.00822151-0.008718903=0.303328587
\end{aligned}$$

and for $\frac{\pi}{2} \leq \arg \alpha^2 \leq \pi$, we have that

$$
\begin{aligned}
V(r,\theta) &\geq \tilde{V}(r,\theta) - \epsilon(r,\theta) - 0.000571171 - 0.000128433 \\
&\geq \tilde{V}(r,\frac{\pi}{2}) - \epsilon(r,\frac{\pi}{2}) - 0.000699604 \\
&\geq \tilde{V}(\frac{\log 3}{2},\frac{\pi}{2}) - \epsilon(\frac{\log 3}{2},\frac{\pi}{2}) - 0.000699604 \\
&\geq 0.279225 - 0.00185844 - 0.000699604 = 0.276666956.
\end{aligned}
$$

□

Theorem 4.11. *The volume of any closed orientable hyperbolic 3-manifold is at least* 0.276666*.*

Proof. In [GMT] it is shown that the shortest geodesic in the manifold has a tube of radius at least 0.529595 or the manifold is Vol3. Further, it is shown that if the tube radius is less than $\frac{\log 3}{2}$ then the geodesic has length at least 1.059536368901. In either of these exceptional cases, the volume is greater than 1.01. □

5 Applications

We have already noted one application, namely Theorem 4.11. We now present applications to the symmetry groups of manifolds. It should be noted that the techniques being employed here are essentially identical to those in [GM98].

We first summarize the relevant results from [GM98].

Proposition 5.1. *Let Γ be a Kleinian group containing a simple element of order p with the only primitive torsion in Γ also of order p. If*

i) $p = 4$, then either Γ is arithmetic or there is a tube of radius at least 0.6130 *in $\mathbb{H}^3/\Gamma$.*

ii) $p = 5$, then there is a tube of radius at least 0.626 *in $\mathbb{H}^3/\Gamma$.*

iii) $p = 6$, then there is a tube of radius at least 0.658

iv) $p \geq 7$, then there is a tube of radius at least $r_p = \cosh^{-1}\left(\frac{1}{2\sin(\pi/p)}\right)$.

For the sake of accuracy we note that as presented in [GM98], there are other possibilities if $p = 4$ or $p = 5$, but those possibilities contradict our assumption that the only primitive torsion has order p. Thus we omit them.

We now provide an improved version of a result in [GM98].

Theorem 5.2. *Let M be a closed orientable hyperbolic 3-manifold which has an orientation preserving symmetry of order $p \geq 4$. If the group generated by this symmetry has nonempty fixed point set, then*

$$Vol(M) \geq \begin{cases} 0.9427 & \text{if } p = 4 \\ 1.69427 & \text{if } p = 5 \\ 2.18088 & \text{if } p = 6 \\ p \inf\limits_{r \geq r_p} V(r,\theta) & \text{if } p \geq 7. \end{cases}$$

Proof. Let f be a lift to $\mathbb{H}^3$ of the given symmetry. Then $\Gamma =< \pi_1(M), f >$ is a Kleinian group, f is a simple element of Γ and the only primitive torsion is of order p. So $\mathrm{Vol}(M) \geq p \cdot \mathrm{Vol}(\mathbb{H}^3/\Gamma)$. At this point, we need only trace through the various cases of Proposition 5.1. If $p \geq 6$, then the result is obvious. If $p = 4$, then there are two possibilities. If Γ is arithmetic, then M is arithmetic. Since the Weeks manifold is known to be the smallest volume arithmetic manifold, $\mathrm{Vol}(M) \geq 0.9427\ldots$. If $\mathbb{H}^3/\Gamma$ contains a tube of radius at least 0.6130, then $\mathrm{Vol}(M) \geq 4 \cdot 0.328606$. Likewise, if $p = 5$, then $\mathbb{H}^3/\Gamma$ contains a tube of radius at least 0.626 in which case $\mathrm{Vol}(M) \geq 5 \cdot 0.338854$. ☐

This result required that the group generated by the symmetry has fixed points. However, it is easy to develop a result which drops that restriction and replaces it with a restriction on the order.

Theorem 5.3. *Let M be a closed orientable hyperbolic 3-manifold which has an orientation preserving symmetry of prime order $p > 3$. Then $Vol(M) \geq p \cdot 0.276$.*

Proof. There are two cases to consider, depending on whether the symmetry has fixed points. If the symmetry does have fixed points, then any power of it has the same fixed point set so we may apply the previous theorem. This yields results strictly better than what we are seeking. If the symmetry does not have fixed points, then the group it generates will act without fixed points. We may then consider a new manifold M' given by M modulo the action of the symmetry. By Theorem 4.11, $\mathrm{Vol}(M') \geq 0.276$ and the result follows. ☐

This result allows us to place restrictions on the order of the symmetry group of the smallest hyperbolic 3-manifold.

Corollary 5.4. *The order of the symmetry group of the smallest volume hyperbolic 3-manifold is of the form* $2^m 3^n$ *for some* $m, n \geq 0$.

Proof. If a prime p divides the order of the symmetry group, then there is a symmetry of order p. Thus $\mathrm{Vol}(M) \geq p \cdot 0.276$. If $p > 4$ then M has volume greater than that of the Weeks manifold. □

The symmetry group of the Weeks manifold has order 12, so it is likely that no other primes can be excluded.

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