A UNIVERSAL UPPER BOUND ON DENSITY OF TUBE PACKINGS IN HYPERBOLIC SPACE

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Abstract. Using techniques developed to solve the corresponding Euclidean problem, we produce a universal upper bound on the density of a packing of tubes in $\mathbb{H}^3$.

1. Introduction

In $\mathbb{E}^3$, it is rather intuitive that the densest way to pack congruent infinite right circular cylinders is to have them all parallel to each other and arranged using the hexagonal packing of circles in $\mathbb{E}^2$. The optimal density is then $\frac{\pi}{\sqrt{12}}$. That these statements are true was proved [BK90] by analyzing cross sections of the Dirichlet domain of such a packing.

Our general approach to dealing with the analogous problem in hyperbolic space will be to follow the same steps as Bezdek and Kuperberg [BK90]. However, in $\mathbb{H}^3$, most of the statements from the previous paragraph become meaningless, or at best ambiguous. Thus, we first need to find an appropriate hyperbolization of the problem.

One source of trouble is that Euclidean objects often have multiple equivalent definitions which, when used in hyperbolic space, become nonequivalent. A Euclidean right circular cylinder can be viewed as either the set of all points within a fixed distance of some line, or as the union of all lines passing perpendicularly through some disk. The latter definition corresponds more closely to the more general meaning of the word cylinder, but the former definition tends to be more useful. Thus, we will use a different word, *tube*, to refer to the set of points within a fixed distance of some line, and we will then answer questions about tube packings.

In addition to defining the objects of study, we must define density. In Euclidean space, it is not usually too difficult to define the density of a packing, but in hyperbolic space, this can be difficult, even for disk packings in $\mathbb{H}^2$ [BR03]. To avoid these difficulties, we will define instead an *upper bound on density.*

Definition 1.1. *Given a tube T in a packing, let D be its Dirichlet domain. Choose a point p on the axis of T. For a given positive l, we truncate T and D along planes perpendicular to the axis and at a distance of l from p. We may then compute the density of the truncated tube within the truncated Dirichlet domain. Taking the* $\limsup$ *as l goes to infinity produces an upper* *bound on the density of T within D. An upper bound on the density of the packing is then the supremum over all choices of T and p of the upper bound on the density of T within D (or any number larger than the supremum).*

A somewhat lesser difficulty is that while Euclidean space can be scaled arbitrarily, hyperbolic space can not. Thus, hyperbolic results will likely depend on the radius of the tubes. We will develop an upper bound on density which will depend on the tube radius. There are already various results [MM00, Prz04] of this nature. These prior results tend to be strongest for large tube radii and practically useless for small tube radii. This result, as a generalization of a Euclidean result, will work best for small tube radii. The various results then complement one another and provide a universal upper bound on density.

Still another difficulty is that the optimal Euclidean packing, if simply adapted for hyperbolic space is clearly no longer optimal, leaving us without a candidate for an optimal packing. If the situation is similar to that of packing disks in $\mathbb{H}^n$ [BR03], it's possible that the very concept of an optimal packing is quite complicated. We do not attempt to determine radii for which there is an optimal packing, nor do we attempt to present an optimal packing for any particular radius. It is unlikely that there are any radii for which our result is sharp. In fact, it's not even known whether density must approach zero as the tube radius approaches zero. Our result limits to the Euclidean case in which density is $\frac{\pi}{\sqrt{12}}$, and thus is possibly very far from optimal for small tube radii. Nonetheless, all earlier results limited to 1 for small tube radii, so this is a substantial improvement.

The main result that we prove is:

Theorem 4.3. *The density of a packing of tubes of radius r in $\mathbb{H}^3$ is at most*

$$\frac{\sinh r \sin^{-1} \frac{1}{2\cosh r}}{\sinh^{-1} \frac{\tanh r}{\sqrt{3}}}$$

As a simple consequence, one gets a universal upper bound on density.

In addition we provide applications to hyperbolic manifolds.

2. Parabolas in hyperbolic space

In [BK90], many of the key steps in the proof involve some fairly obvious properties of parabolas. In the hyperbolization of the problem (and solution) one must establish similar properties for the analogous curves in hyperbolic space. One first needs to determine the corresponding curve and then one needs to develop computational tools for dealing with it. As it turns out, the hyperbolic analog is, in the Klein model, a hyperbola.

In [BK90] parabolas appear by determining the curve in some plane which is equidistant from some point (in the plane) and some line (possibly not in the plane). We may assume that the line and point are disjoint. (Actually, the equidistant curve could also be line(s) in some special cases, but we will ignore this for the moment). We will need to produce the analogous curve in hyperbolic space, although, as usual, the hyperbolic computations are significantly more complicated.

We will start in the upper half-space model $\mathbb{C} \times \mathbb{R}^+$, and then switch to other models or coordinate systems as the need develops. Let the (hyperbolic) plane be the (Euclidean) hemisphere of radius 1 centered at $(0,0)$. Let the point be $(0,1)$ and let the line have endpoints $(u,0)$ and $(v,0)$. The line is then the parametrized curve

$$(\frac{u+v}{2} + \frac{u-v}{2}\cos t, |\frac{u-v}{2}| \sin t) \text{ for } t \in (0,\pi).$$

We note that it's also possible for a (hyperbolic) line to be a vertical (Euclidean) line with only one endpoint in $\mathbb{C} \times \{0\}$. However, this case will be ruled out by later considerations.

Points on the plane which are at a distance d from $(0,1)$ are of the form $(e^{i\theta} \tanh d, \operatorname{sech} d)$. If such a point is on the equidistant curve, then the distance to the line must also be d. Hence, there must be exactly one point on the line which is at a distance d from $(e^{i\theta} \tanh d, \operatorname{sech} d)$. Then, we wish to determine whether there is exactly one value of $t \in (0,\pi)$ for which

$$\cosh d = 1 + \frac{|\frac{u+v}{2} + \frac{u-v}{2}\cos t - e^{i\theta}\tanh d|^2 + (|\frac{u-v}{2}|\sin t - \operatorname{sech} d)^2}{2|\frac{u-v}{2}|\sin t \operatorname{sech} d}$$

After some rearrangement, this becomes

$$|u-v| = \frac{|\frac{u+v}{2} - e^{i\theta}\tanh d|^2 + |\frac{u-v}{2}|^2 + \operatorname{sech}^2 d}{\sin t} + \frac{2\mathrm{Re}[(\frac{u+v}{2} - e^{i\theta}\tanh d)\frac{\bar{u}-\bar{v}}{2}]\cos t}{\sin t}$$

This is an equation of the form $a\csc t + b\cot t = c$ with $a > |b|$. One can easily check that such an equation has exactly one solution in $(0,\pi)$ if and only if $a^2 = b^2 + c^2$.

Substituting the values for a, b, and c, we see that

$$\left(\left|\frac{u+v}{2} - e^{i\theta}\tanh d\right|^2 + \left|\frac{u-v}{2}\right|^2 + \operatorname{sech}^2 d\right)^2 = \left(2\mathrm{Re}\left[\left(\frac{u+v}{2} - e^{i\theta}\tanh d\right)\frac{\bar{u}-\bar{v}}{2}\right]\right)^2 + |u-v|^2$$

After a little manipulation, this becomes

$$\begin{aligned} |u-v|^2 &= (|u - e^{i\theta}\tanh d|^2 + \operatorname{sech}^2 d)(|v - e^{i\theta}\tanh d|^2 + \operatorname{sech}^2 d) \\ &= (|u|^2 + 1 - 2\tanh d\,\mathrm{Re}[ue^{-i\theta}])(|v|^2 + 1 - 2\tanh d\,\mathrm{Re}[ve^{-i\theta}]) \end{aligned}$$

Up until now, we've been using the polar coordinates (d,θ) on the hyperbolic plane. At this point, it becomes convenient to change to (x,y) where $x = \tanh d\cos\theta$ and $y = \tanh d\sin\theta$. These are cartesian coordinates for the

Klein model of the hyperbolic plane. Using these coordinates, and letting $u = u_1 + iu_2, v = v_1 + iv_2$, we have

$$(|u|^2 + 1 - 2u_1x - 2u_2y)(|v|^2 + 1 - 2v_1x - 2v_2y) = |u - v|^2$$

which one can readily check is a hyperbola except in some special cases in which we get lines. One can further check that at the center point of this hyperbola, the left side of the equation is less than the right side. At the point $(0,0)$, the reverse is true, unless $u\bar{v} = -1$ which would violate our assumption about the line and point being disjoint. This implies that $(0,0)$ is on the convex side of the hyperbola. The convexity here is in the Klein model, but since lines in hyperbolic space are also lines in the Klein model, we see that the convexity is in fact true in the hyperbolic plane.

3. Minimizing Area

An upper bound on density is defined via the Dirichlet domain for a given tube in a packing. Following [BK90] we define a *Dirichlet domain slice* to be any cross section of the Dirichlet domain by a plane perpendicular to the axis of the tube. The *center* of the Dirichlet domain slice is the point which lies on the axis of the tube. Once we introduce coordinates, we will usually place this point at the origin.

From here on, we will assume that we are given a specific packing of tubes of radius r.

First, we develop a relationship between the slice and an upper bound on density for the packing.

Proposition 3.1. $\frac{\pi \sinh^2 r}{\min\limits_S \int\limits_S \cosh R\, dA}$ *is an upper bound on density, where S varies over all possible slices and R is the distance to the center of the slice.*

Proof. We will use cylindrical coordinates (R, θ, z) on $\mathbb{H}^3$. Specifically, choose a half-plane P and call its edge ℓ. Given a point $p \in \mathbb{H}^3$, let p' be the orthogonal projection of p onto ℓ. Let R be the distance from p to p', θ be the angle between the segment pp' and P (with some suitably chosen sign convention), and let z be the signed distance from p' to some chosen basepoint z_0 on ℓ. For our purposes, the only restriction is that ℓ be the axis of a tube T in the packing.

An upper bound on density is then defined as any number as large as

$$\max_{T,z_0} \limsup_{l\to\infty} \frac{\mathrm{Vol}(T \cap \{(R,\theta,z) : |z| \le l\})}{\mathrm{Vol}(D \cap \{(R,\theta,z) : |z| \le l\})}.$$

Let S_z be the slice of the Dirichlet domain D at height z. Then the upper bound may be expressed as

$$\max_{T,z_0} \limsup_{l\to\infty} \frac{2\pi l \sinh^2 r}{\int_{-l}^{l} \int\limits_{S_z} \cosh R\, dA dz} \le \frac{2\pi l \sinh^2 r}{\min\limits_S 2l \int\limits_S \cosh R\, dA}.$$

□

To find an upper bound on density, it will be sufficient to find a lower bound on $\int_S \cosh R\, dA$, a certain weighted area of the slice.

Proposition 3.2. *The Dirichlet domain slice is bounded by lines and Klein model hyperbolas of the type described in the previous section. The slice is convex.*

Proof. A point is on the boundary of the slice if it is equidistant from the center and the axis of some other tube. Thus the boundary consists of the hyperbolas described earlier, or when the hyperbolas degenerate, lines. We had shown earlier that the origin is on the convex side of these curves. □

The points at which these various curves intersect will be called vertices of the slice. They are cross sections of edges of the Dirichlet domain.

Proposition 3.3. *The distance from any vertex to the center of the slice is at least* $\sinh^{-1} \frac{2\sinh r}{\sqrt{3}}$.

Proof. The argument in [BK91] can be copied practically verbatim, although the computations must be done hyperbolically, to show that if a sphere in $\mathbb{H}^3$ intersects three disjoint tubes of radius r, then the radius of the sphere is at least $\sinh^{-1} \frac{2\sinh r}{\sqrt{3}} - r$. Since the vertices are equidistant from at least three tubes, we see that any vertex must be at least $\sinh^{-1} \frac{2\sinh r}{\sqrt{3}}$ away from the axes of the tubes. □

Later, we will use this result to truncate the slice, discarding everything that is farther from the center than $\sinh^{-1} \frac{2\sinh r}{\sqrt{3}}$. However, we also need to know how far vertices can be from each other.

Proposition 3.4. *Choose two points P_1 and P_2 on the boundary of the slice which are at a distance of* $\sinh^{-1} \frac{2\sinh r}{\sqrt{3}}$ *from the center and have no vertex between them. If the points are connected by a line, then the angular separation (based at the center) between the points is at most* $2\sin^{-1} \frac{1}{2\cosh r}$. *If the points are connected by a hyperbola, the angular separation is at most* $2\cos^{-1}\left(\frac{\cosh r\sqrt{3+4\sinh^2 r}-1}{\cosh 2r}\right)$.

Proof. It's easy to verify that if a triangle has two equal sides of length $\sinh^{-1} \frac{2\sinh r}{\sqrt{3}}$ and the altitude has length at least r then the angle (between the two equal length sides) is at most $2\sin^{-1} \frac{1}{2\cosh r}$.

In the case of a hyperbola, we have more work to do. Let the tube whose Dirichlet domain we're using be T_1. The hyperbola under consideration is equidistant from the axis of T_1 and the axis of some other tube T_2. Consider the plane which contains the axis of T_2 and is perpendicular to the common perpendicular between the axes of T_1 and T_2. The points P_1 and P_2 are at a distance of $\sinh^{-1} \frac{2\sinh r}{\sqrt{3}}$ from the axis of T_2 so are within this distance of the plane. Thus we are dealing with the following situation: a line segment BC of unknown length, and two other line segments AB and CP_1 intersecting BC perpendicularly (and on the same side of BC) where the length of AB is at least $2r$, the length of AP_1 is $\sinh^{-1} \frac{2\sinh r}{\sqrt{3}}$, and the length of CP_1 is at most $\sinh^{-1} \frac{2\sinh r}{\sqrt{3}}$. The segment AB joins the center A of the slice to the constructed plane, the segment BC lies within the plane, and the other two segments connect P_1 to A and the plane. We wish to know how large $\angle BAP_1$ can be (and then double the result). The Laws of Cosines then indicates that $\cosh BP_1 = \cosh AB \cosh AP_1 - \sinh AB \sinh AP_1 \cos \angle BAP_1$ and $\cos \angle ABP_1 = \frac{\cosh AB \cosh BP_1 - \cosh AP_1}{\sinh AB \sinh BP_1}$. The Law of Sines further tells us that $\cos \angle ABP_1 = \sin \angle CBP_1 = \frac{\sinh CP_1}{\sinh BP_1}$. Some straightforward algebra then yields $\cos \angle BAP_1 = \frac{\sinh AB \cosh AP_1 - \sinh CP_1}{\cosh AB \sinh AP_1}$. One can readily check that this expression is increasing in AB and decreasing in CP_1. Thus $\cos \angle BAP_1$ is minimized when AB has length $2r$ and CP_1 has length $\sinh^{-1} \frac{2\sinh r}{\sqrt{3}}$, yielding the desired result. □

We are going to cut the Dirichlet domain slice along specific rays originating at the center, thereby breaking it up into various pieces bounded by two radial segments and either a line or a hyperbola. We'll refer to these pieces as *sectors* of the slice. Of course, given two radial segments of specified lengths, there is only one line joining their endpoints. However, the same two radial segments could be joined by many different hyperbolas. Thus we will need to find the hyperbola which minimizes the weighted area of the sector.

Proposition 3.5. *Let P_1 and P_2 be two points which are at a distance of $\sinh^{-1} \frac{2\sinh r}{\sqrt{3}}$ from the center, and are also on the boundary of the slice. Suppose further that the angle between P_1 and P_2 is greater than $2\sin^{-1} \frac{1}{2\cosh r}$ and at most $2\cos^{-1}\left(\frac{\cosh r\sqrt{3+4\sinh^2 r}-1}{\cosh 2r}\right)$. Then among all hyperbolas of the type discussed earlier which join these points while staying at least r from the center, the one which minimizes the weighted area of the sector is the one which is tangent to circle of radius r at the point halfway between P_1 and P_2.*

Proof. The curves with which we are dealing are specific types of hyperbolas in the Klein model. A general hyperbola would satisfy an equation of the form $ax^2 + bxy + cy^2 + dx + ey + f = 0$ where $b^2 - 4ac > 0$. Accounting for choice of scale, there are five parameters present, whereas we earlier generated a four-parameter family of hyperbolas (or two complex parameters), which we shall refer to as *allowable* hyperbolas. Rather than attempt to remain within this four-parameter family, we will briefly enter the realm of all possible hyperbolas, to simplify some of the computations.

First, we shall show that given an allowable hyperbola meeting the hypotheses, there is a hyperbola which is symmetric about the x-axis, goes through the given points, and bounds no more weighted area than the original. However, this hyperbola will in general not be allowable. Thereafter, we find a hyperbola which is more likely to be allowable, goes through the given points, is symmetric about the x-axis, and does not increase weighted area. However, it is not immediately obvious that this hyperbola does not get too close to the origin. We then verify that among such hyperbolas, the one which passes through $(\tanh r, 0)$ bounds the least weighted area and is closest to the origin at $(\tanh r, 0)$ and that this hyperbola is allowable. This would complete the proof of the theorem.

We start with an allowable hyperbola $ax^2+bxy+cy^2+dx+ey+f=0$. Being allowable means that there is some choice of real parameters u_1, u_2, v_1, v_2, with

$$\begin{aligned}
a &= 4u_1v_1 \\
b &= 4(u_1v_2+u_2v_1) \\
c &= 4u_2v_2 \\
d &= -2(u_1(v_1^2+v_2^2+1)+v_1(u_1^2+u_2^2+1)) \\
e &= -2(u_2(v_1^2+v_2^2+1)+v_2(u_1^2+u_2^2+1)) \\
f &= (u_1^2+u_2^2+1)(v_1^2+v_2^2+1)-(u_1-v_1)^2-(u_2-v_2)^2
\end{aligned}$$

Note that this forces $b^2-4ac \geq 0$ and that if $b^2-4ac=0$ then the curve is not a hyperbola, but rather lines. Also, note that $(e^2-4cf)-(b^2-4ac) = 4(v_2(u_1^2+u_2^2-1)+u_2(1-v_1^2-v_2^2))^2 \geq 0$ and thus that $e^2-4cf \geq b^2-4ac$.

Working in the Klein model again, we may choose the coordinate system so as to have the two given points P_1 and P_2 be symmetric about the x-axis, and have positive x-coordinates. Let the two points be $(x_0, \pm y_0)$ where $x_0 > 0$ and $x_0^2+y_0^2 = \tanh^2 \sinh^{-1} \frac{2\sinh r}{\sqrt{3}}$. Based on the hypotheses, we may further assume that $x_0 < \tanh r$. Having the hyperbola pass through two points which are symmetric about the x-axis does not necessarily make the hyperbola symmetric about the x-axis (i.e $b=e=0$). However, if the hyperbola is not symmetric about the x-axis, then $x_0 = -\frac{e}{b}$ and $c \neq 0$. If we perform the affine transformation $(x,y) \to (x, y+\frac{bx+e}{2c})$, we will get the hyperbola

$$(b^2-4ac)x^2-4c^2y^2+(2be-4cd)x+(e^2-4cf)=0$$

Note the this transformation does not change the (Klein) distance between two points with the same x-coordinate. Thus the vertical extent of the hyperbola at a given x-coordinate will be preserved, and will now be centered about $y=0$. Since the weighted area element in the Klein model is $\cosh R\, dA = \frac{1}{\sqrt{1-x^2-y^2}} \cdot \frac{dx\,dy}{(1-x^2-y^2)^{3/2}} = \frac{dx\,dy}{(1-x^2-y^2)^2}$, this verifies that the resulting symmetric hyperbola bounds less weighted (hyperbolic) area than the original hyperbola. It's not clear that the modified hyperbola stays far enough away from the center or that it's allowable, but this hyperbola is only an intermediate step. Our final hyperbola will be meet the necessary criteria.

Our goal now is to reimpose hypotheses, one by one, without increasing weighted area. First, we take a step toward allowability. There are various ways that an allowable hyperbola can be symmetric about the x-axis, but since we need only find any such hyperbola, we will choose one type, specifically those generated by $uv = 1$. One can readily check that a necessary condition for this is having the x^2 coefficient equal to the constant term. Requiring, in addition, that the x coefficient not be too close to zero would create a sufficient condition. However, we won't bother with this second condition. Thus, the hyperbolas we'll be generating are possibly not allowable, but they are closer to being allowable than the previous hyperbola was.

Let the vertex of the hyperbola be $(x_1, 0)$. Consider the family of symmetric hyperbolas which pass through the points $(x_0, \pm y_0)$ and $(x_1, 0)$. This is a one parameter family

$$x^2 - \frac{(x_1 - x_0)(t - x_0 x_1)}{x_1 y_0^2} y^2 - \frac{x_1^2 + t}{x_1} x + t = 0$$

for $t > x_1^2 > x_0 x_1$. One can readily check that increasing t results in a hyperbola which bounds a larger set. Recall that for an allowable hyperbola, $t = 1$. Since our current value of t is $\frac{e^2 - 4cf}{b^2 - 4ac} \geq 1 > x_1^2$, the symmetric hyperbola we discussed earlier can be replaced by a symmetric hyperbola which passes through the points $(x_0, \pm y_0)$ and $(x_1, 0)$, meets a necessary condition for allowability, and bounds no more weighted area than before. Again, it is not immediately obvious that this new hyperbola does not get too close to the origin.

At this point, we are dealing with hyperbolas of the form

$$x^2 - \frac{(x_1 - x_0)(1 - x_0 x_1)}{x_1 y_0^2} y^2 - \frac{x_1^2 + 1}{x_1} x + 1 = 0$$

Note that $x_1 \geq \tanh r$ as x_1 is the x-coordinate of the vertex of the original hyperbola which was to the right of $(\tanh r, 0)$. Note further, that decreasing x_1 results in a hyperbola which bounds a smaller set and yet still satisfies all of the earlier conditions. Thus, we might as well assume that $x_1 = \tanh r$. In this case, it is easy to check that the hyperbola is allowable. All that now remains of the proof is to show that the vertex of this hyperbola is the point which is closest to the origin. For a hyperbola, if the vertex minimizes distance locally then it minimizes distance globally. One can easily check that the vertex minimizes distance locally if $y_0^2(1 - x_1^2) - 2x_1(x_1 - x_0)(1 - x_0 x_1) > 0$. Keep in mind that x_1 and $x_0^2 + y_0^2$ depend on nothing but r. Thus we may regard this expression as a quadratic in x_0. The coefficient of x_0^2 is negative so the expression will be minimized at one of the extreme values for x_0. We know that

$$\frac{2\sinh r}{\sqrt{3+4\sinh^2 r}}\frac{\cosh r\sqrt{3+4\sinh^2 r}-1}{\cosh 2r} \leq x_0 \leq x_1$$

One can readily check that the expression is positive at either of these points. $\square$

4. Optimizing the Dirichlet domain slice

Recall that the Dirichlet domain slice is a convex region which if represented in the Klein model is bounded by lines and hyperbolas. The vertices of this region lie on or outside the circle of (Klein) radius $\frac{2\sinh r}{\sqrt{3+4\sinh^2 r}}$. Let $Q_1, Q_2, \ldots Q_n$ be the points at which the edges of the slice intersect this circle. By removing any parts of the slice which lie outside of the circle, we will not increase the weighted area of the slice. Parts of the resulting boundary could be arcs. We may introduce new vertices on these arcs and then replace the arcs with chords joining these vertices, further reducing weighted area. If two of the vertices are separated by an angle of at most $2\sin^{-1}\frac{1}{2\cosh r}$ then by removing any region on the far side of the chord joining these points, we will not increase the weighted area. Lastly, if two of the vertices are separated by an angle between $2\sin^{-1}\frac{1}{2\cosh r}$ and $2\cos^{-1}\left(\frac{\cosh r\sqrt{3+4\sinh^2 r}-1}{\cosh 2r}\right)$, then they are joined by a hyperbola. We may replace this hyperbola with the allowable hyperbola which passes through these two vertices and is tangent to the circle of radius $\tanh r$ at a point halfway in between the vertices.

As a result of this process, we see that the Dirichlet domain slice which minimizes weighted area is a union of isosceles triangles of side length $\frac{2\sinh r}{\sqrt{3+4\sinh^2 r}}$ and of specific types of hyperbolic sectors. The weighted area of any such sector is completely determined by the angle it subtends at the origin.

Proposition 4.1. *If the sector is an isosceles triangle with (Klein) side length* $\frac{2\sinh r}{\sqrt{3+4\sinh^2 r}}$ *and vertex angle* $\theta \leq 2\sin^{-1}\frac{1}{2\cosh r}$, *then the weighted area is*

$$A_1(\theta) = \frac{2\sinh r\cos\frac{\theta}{2}\sinh^{-1}\frac{2\sin\frac{\theta}{2}\sinh r}{\sqrt{3}}}{\sqrt{3+4\sin^2\frac{\theta}{2}\sinh^2 r}}$$

Proof. Using polar coordinates (ρ, ϕ) on the Klein model, the weighted area of the triangle Δ is

$$\begin{aligned}
\int_\Delta \cosh R\, dA &= 2\int_0^{\frac{\theta}{2}} \int_0^{\frac{2\sinh r \cos\frac{\theta}{2}}{\cos\phi\sqrt{3+4\sinh^2 r}}} \frac{\rho}{(1-\rho^2)^2}\, d\rho d\phi \\
&= \int_0^{\frac{\theta}{2}} \frac{1}{1-\frac{4\sinh^2 r\cos^2\frac{\theta}{2}}{\cos^2\phi(3+4\sinh^2 r)}} - 1\, d\phi \\
&= \frac{4\sinh^2 r\cos^2\frac{\theta}{2}}{3+4\sinh^2 r}\int_0^{\frac{\theta}{2}} \frac{1}{\cos^2\phi - \frac{4\sinh^2 r\cos^2\frac{\theta}{2}}{3+4\sinh^2 r}}\, d\phi \\
&= \frac{1}{\sqrt{\frac{3+4\sinh^2 r}{4\sinh^2 r\cos^2\frac{\theta}{2}} - 1}} \tanh^{-1} \frac{\tan\frac{\theta}{2}}{\sqrt{\frac{3+4\sinh^2 r}{4\sinh^2 r\cos^2\frac{\theta}{2}} - 1}} \\
&= \frac{2\sinh r\cos\frac{\theta}{2}\sinh^{-1}\frac{2\sin\frac{\theta}{2}\sinh r}{\sqrt{3}}}{\sqrt{3+4\sinh^2 r\sin^2\frac{\theta}{2}}}
\end{aligned}$$

□

If the sector is bounded by a hyperbola, and has a vertex angle $\theta > 2\sin^{-1}\frac{1}{2\cosh r}$ then the weighted area is

$$A_2(\theta) = 2\int_0^{\frac{\theta}{2}} \int_0^{f(\theta,\phi)} \frac{\rho}{(1-\rho^2)^2}\, d\rho\, d\phi$$

where $\rho = f(\theta, \phi)$ describes the hyperbola in polar coordinates (ρ, ϕ) on the Klein model.

Note that A_1 and A_2 are actually functions of both r and θ, but we are suppressing the r dependence.

Proposition 4.2. *Both A_1 and A_2 are concave down. Thus the density of a sector of a circle of vertex angle θ in one of these types of sectors is maximized at $\theta = 2\sin^{-1}\frac{1}{2\cosh r}$ or $\theta = 2\cos^{-1}\left(\frac{\cosh r\sqrt{3+4\sinh^2 r}-1}{\cosh 2r}\right)$.*

Proof. To check that A_1 is concave down, we note that it can be written as

$$A_1(\theta) = \frac{\partial}{\partial\theta}\left(\left(\sinh^{-1}\left(\frac{2\sin\frac{\theta}{2}\sinh r}{\sqrt{3}}\right)\right)^2\right)$$

Then $A_1(\theta) = \frac{\partial}{\partial\theta}(h^2(\theta))$ for some function $h(\theta)$, so $A_1''(\theta) = 6h'h'' + 2hh'''$. Some simple computation verifies that $h > 0$, $h' > 0$, $h'' < 0$, and $h''' < 0$, so A_1 is concave down as a function of θ.

Checking that A_2 is concave down is more complicated. First, we rearrange the formulation of A_2 to facilitate this computation.

$$\begin{aligned} A_2(\theta) &= 2\int_0^{\tanh r}\int_0^{\frac{\theta}{2}} \frac{\rho}{(1-\rho^2)^2}\,d\phi\,d\rho + \\ & \quad 2\int_{\tanh r}^{\frac{2\sinh r}{\sqrt{3+4\sinh^2 r}}}\int_{g(\theta,\rho)}^{\frac{\theta}{2}} \frac{\rho}{(1-\rho^2)^2}\,d\phi\,d\rho \end{aligned}$$

where $\phi = g(\theta,\rho)$ describes the hyperbola in polar coordinates.

The first term is linear in θ so will not affect concavity. The second term will be concave down if $g(\theta,\rho)$ can be shown to be concave up as a function of θ. Although it requires a fair amount of computation, this is not particularly difficult.

The weighted density of a sector of a circle in one of these regions will be $\frac{\theta\sinh^2 r}{A_i(\theta)}$ where i would be 1 or 2, depending on the size of θ. As A_i is concave down, it is easy to check that the density is maximized at one of the endpoints and that the endpoint $\theta = 0$ is not the maximum. $\square$

One can check that of these, the larger density is achieved when $\theta = 2\sin^{-1}\frac{1}{2\cosh r}$ and thus we have

Theorem 4.3. *The density of a packing of tubes of radius r in $\mathbb{H}^3$ is at most*

$$\frac{\sinh r \sin^{-1}\frac{1}{2\cosh r}}{\sinh^{-1}\frac{\tanh r}{\sqrt{3}}}$$

We note that one could actually improve this result slightly. We could find the smallest weighted area Dirichlet domain slice, using the concavity of the area functions. Such a Dirichlet domain slice would have to be composed of sectors of vertex angle $2\sin^{-1}\frac{1}{2\cosh r}$, vertex angle $2\cos^{-1}\left(\frac{\cosh r\sqrt{3+4\sinh^2 r}-1}{\cosh 2r}\right)$, and at most one sector of some other vertex angle. There are then only finitely many cases one must check to determine the optimum density. However, the improvement is very small and significantly complicates the statement of the result. Thus we do not pursue this course of action.

Corollary 4.4. *The density of a symmetric packing of tubes in hyperbolic space is at most* 0.91.

Proof. If the tube radius is at most 1.2 then the density bound we just developed will be at most 0.91. If the tube radius is 1.2 or more, then the density bound in [Prz04] is at most 0.91. $\square$

Of course, one could also produce a slightly stronger, although more complicated result by incorporating the three current upper bounds on density: this one for $r \leq 1.2$, our earlier result [Prz04] for $1.2 \leq r \leq 7.1$, and Marshall and Martin's [MM00] for $r \geq 7.1$. We let $\rho(r)$ denote the upper bound on density achieved in this fashion.

5. Applications

The primary application of a result such as ours is to the study of hyperbolic 3-manifolds. Typically, one tries to locate a geodesic in the manifold such that the maximal tube about this geodesic has certain desirable properties, such as a lower bound on tube radius. The goal is usually to use this information about the tube to produce a lower bound on the tube volume, and hence a lower bound on the manifold volume. Our upper bound on tube density will allow stronger lower bounds on manifold volume.

First, we need some preliminaries.

Theorem 5.1. [GMT03] *The orientable hyperbolic 3-manifold of minimal volume has a tube of radius at least* $\frac{\log 3}{2}$ *about its shortest geodesic.*

Theorem 5.2. [Ago02] *Let* M *be a hyperbolic 3-manifold and let* γ *be a geodesic link in* M *with an embedded open tubular neighborhood* T *of radius* r*. Let* M_γ *denote* $M \setminus \gamma$ *in a complete hyperbolic metric. Then*

$$\mathrm{Vol}(M_\gamma) \leq (\coth r \coth 2r)^{\frac{3}{2}}(\mathrm{Vol}(M) + (\frac{\coth r}{\coth 2r} - 1)\mathrm{Vol}(T))$$

We may improve Agol's result [Ago02] by incorporating information about tube density.

Corollary 5.3. *Let* M *be a hyperbolic 3-manifold and let* γ *be a geodesic link in* M *with an embedded open tubular neighborhood* T *of radius* r*. Let* M_γ *denote* $M \setminus \gamma$ *in a complete hyperbolic metric. Then*

$$\mathrm{Vol}(M_\gamma) \leq (\coth r \coth 2r)^{\frac{3}{2}}\mathrm{Vol}(M)(1 + \rho(r)(\frac{\coth r}{\coth 2r} - 1))$$

Proof. One simply notices that $\mathrm{Vol}(T) \leq \rho(r)\mathrm{Vol}(M)$. □

We note that as $\rho(r)$ is at most 0.91, one could simplify the corollary a bit by replacing $\rho(r)$ with 0.91. However, this would weaken the result, particularly when r is either very large or very small.

This corollary may now be used to improve the current lower bound on the volume of hyperbolic 3-manifolds.

Proposition 5.4. *All orientable hyperbolic 3-manifolds have volume at least* 0.3324

Proof. It is known [CM01] that the minimal volume noncompact orientable hyperbolic 3-manifold has volume 2.0298.... As stated earlier, the minimal volume orientable hyperbolic 3-manifold has a tube of radius at least $\frac{\log 3}{2}$ about some geodesic. Inserting these two statements into Corollary 5.3 produces the desired lower bound. □

Finally, we include a chart which helps to indicate the extent to which the current density results fall short. Using Oliver Goodman's program `tube` [Goo], we computed the radii of the maximal tubes about geodesics of length less than 3 in the first (roughly) 1400 manifolds in the Weeks census [Wee](a few manifolds or geodesics caused trouble for `tube` so were omitted). The resulting data are displayed in Figure 1.

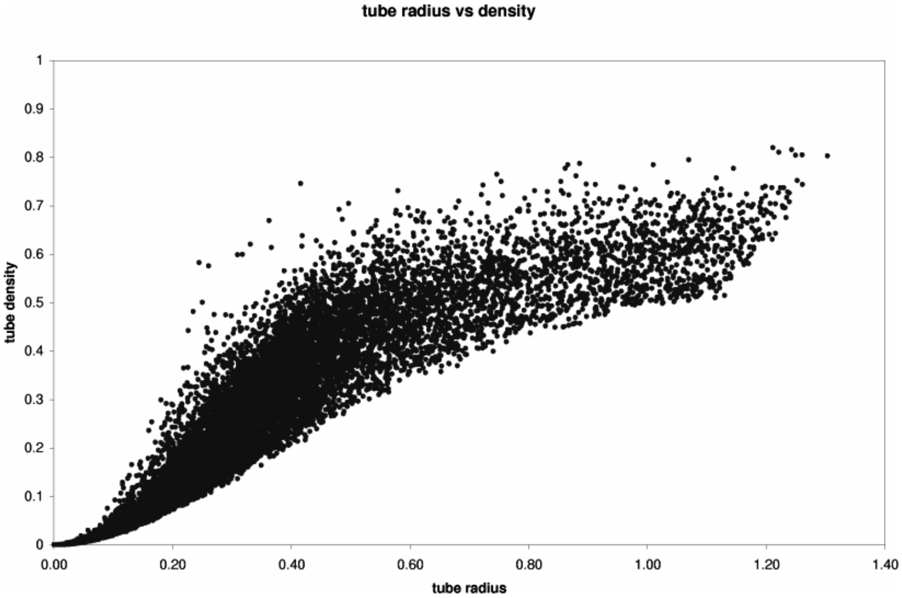

FIGURE 1

We offer a few comments on this illustration. First, the current upper bounds on density are at least 0.85, regardless of radius, indicating that there's still a long way to go, particularly for small tube radii. Second, the chart could be perceived as suggesting that there is a lower bound on density. This is unlikely. Rather, there are known lower bounds on tube volume [GMM01, MM03]. Combined with our using only relatively small volume manifolds, one gets an illusory lower bound on density. Finally, the data for tubes of very small radius are biased. In order for a very small radius tube to have anything other than a very small volume, it would have to be very long. By dealing with only geodesics of length less than 3, we have made it impossible to use this chart to discern information about the density of very small radius tubes.

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