# PACKING DISKS ON A TORUS

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ABSTRACT. We determine the densest packing of two congruent disks on a torus. The maximal density varies depending on the ratio of the disk radius to the length of a closed geodesic on the torus.

## 1. Introduction

Ball packing is one of the classic topics in mathematics. Of course, many of the simple questions were answered long ago, leaving only the hard questions and the overlooked questions. Here, we address what we believe to be one of the overlooked questions which has recently gained some relevance in the study of hyperbolic 3-manifolds.

We will investigate a specific kind of disk packing on a torus. A cylinder covers a torus, so a disk packing on a torus may be lifted to a disk packing on a cylinder. The cylindrical problem has previously been considered. See [FT62] and [BFT64].

A torus may be regarded geometrically as a quotient $\mathbb{R}^2/\Lambda$ of the Euclidean plane by a rank two lattice $\Lambda$. However, the specifics of the global geometry depend on the lattice $\Lambda$. The local geometry of the Euclidean plane passes down to the torus, so we may talk about disks within the torus, though if the radius is too large, the disk would not embed. One simple packing problem would be to find the largest radius disk that may be placed within a torus of a given area, or conversely, to find the smallest area torus that can contain a disk of a given radius. The ratio of the area of the disk to the area of the torus is called the density of the packing. The solution to each of the above problems is based on the hexagonal packing of disks in the plane, which is the maximum density disk packing in the plane. Letting $\Lambda$ be the corresponding lattice, one constructs a torus on which it is possible for disk packing to achieve the same density as the hexagonal disk packing does in the plane, $\frac{\pi}{\sqrt{12}}$.

The problem becomes more interesting when we impose restrictions on the geometry of the torus. The restriction we will consider here is that the torus has a closed geodesic of length 1. This corresponds to requiring that $\Lambda$ contain a primitive element of length 1. For some disk radii, like $\frac{1}{2}$, the hexagonal packing will still satisfy this restriction, but for most disk radii, the hexagonal packing is no longer possible.

2000 *Mathematics Subject Classification.* Primary 52C15; Secondary 11H31.

In the study of hyperbolic 3-manifolds, one often deals with simple closed geodesics. About such a geodesic, one can find an embedded tube (topologically a solid torus, geometrically the set of all points within some fixed distance of the geodesic). The boundary of such a tube is a torus which happens to bear the usual Euclidean geometry. Except in some elementary situations, there will be a maximal radius for such a tube. If we choose a maximal radius embedded tube, then there will be a point on its boundary at which the tube abuts itself. In fact, this point is actually two boundary points which have become identified. Based on various geometric considerations, one can find an object (ideally a convex object, or even a disk) on the tube boundary about each of these points. One then wants to use the area of this object to place a lower bound on the area of the tube boundary, which would place a lower bound on the tube's volume, which would provide geometric information about the manifold. See [GM98], [Prz03], [GMM01], and [MM03].

Accordingly, we ask the following question:

**Question 1.** *Given $r > 0$, among all tori containing a closed geodesic of length 1, what is the smallest possible area for a torus which contains two nonoverlapping disks of radius $r$?*

In order to answer this question, we will need to determine both the lattice (i.e. the shape of the torus) and how the two disks are positioned on the torus relative to each other. The key result (minus the details) is:

**Theorem 2.11.** *Depending on the disks' radius, the optimal packing is of one of two types, called optimal lattice packings and equilateral packings.*

For each of these two types of packings, we then determine the maximal density.

**Theorem 3.1.** *For optimal lattice packings of disks of radius $r$, the maximal density is $\max\limits_{|f(m,n,r)|\leq\frac{1}{2}} \frac{\pi}{4\sqrt{1-f^2(m,n,r)}}$ where $f(m,n,r) = \frac{\frac{1}{4r^2}-2m^2-2n^2}{2(m^2-n^2)}$ and $m$ and $n$ are relatively prime integers. There is an exceptional case when $r = \frac{1}{4}$. In this case, a hexagonal packing is possible, with density $\frac{\pi}{\sqrt{12}}$.*

**Theorem 4.1.** *For equilateral packings of disks of radius $r$, the maximal density is $\max\limits_{g_\pm(m,n,r)\geq 2\sqrt{3}} \frac{\pi}{g_\pm(m,n,r)}$ where $\gcd(m,n) = 1$ and*

$$g_\pm(m,n,r) = \frac{(m^2+mn+\frac{1}{4r^2})\sqrt{3}}{2(n^2+m^2+mn)} \pm (n+2m)\frac{\sqrt{\frac{n^2}{r^2}-(m(m+n)-\frac{1}{4r^2})^2}}{2(n^3+m^2n+mn^2)}.$$

*There is one exception. If $r = \frac{1}{2}$, a hexagonal packing is possible.*

These upper bounds are, unfortunately, not expressed in a simple form. However, their computation is a finite problem (and not even a particularly large one), and thus easily handled by a computer.

## 2. Locating the Optimal Packings

Although we stated the problem in terms of tori, it is more convenient to work in the Euclidean plane. When we lift the torus to its universal cover, the Euclidean plane, each disk on the torus will lift to a lattice packing of disks. Thus, for the two disks, we will have two offset lattice packings of nonoverlapping congruent disks. The set of disk centers for such a packing will be called a double-lattice.

**Definition 2.1.** *Given a rank 2 lattice $\Lambda$, a $\Lambda$-double lattice consists of all points in $\Lambda \cup (\mathbf{w} + \Lambda)$ for some choice $\mathbf{w} \notin \Lambda$. A double lattice is a $\Lambda$-double lattice, for any choice of $\Lambda$. Placing congruent nonoverlapping disks centered at the points of a double lattice results in a double lattice packing of disks.*

We must also define the density of such a packing.

**Definition 2.2.** *Given a $\Lambda$-double lattice packing of disks, choose a fundamental domain $D$ for $\Lambda$. Further, find the subset $E$ of $D$ consisting of all points in $D$ which lie in any of the disks. The density of the packing is $\frac{Area(E)}{Area(D)}$.*

With these definitions, we may now restate Question 1.

**Question 2.** *For disks of radius $r > 0$, among all double lattice packings for which the lattice contains a primitive element of length 1, what is the largest possible density?*

What we want to do is to determine an upper bound on density provided we have information about the radius of the disks in the packing. However, it will be easier to start out by reversing the question. Thus, we first ask the following question:

**Question 3.** *Given a specific lattice $\Lambda$, what is the largest $r$ for which nonoverlapping disks of radius $r$ may be centered at the points of a $\Lambda$-double lattice?*

Changing the scale of the double lattice will simply change $r$ by the same proportionality factor. It is convenient to scale the double lattice so as to make 1 a primitive element. Further, it is notationally convenient to regard the Euclidean plane as $\mathbb{C}$.

**Theorem 2.3.** *Given a lattice $\Lambda$, by scaling and rotating if necessary, we may assume that one of the generators is 1 and the other is a number $\mathbf{z}$ satisfying $|\mathrm{Re}(\mathbf{z})| \leq \frac{1}{2}$, $|\mathbf{z}| \geq 1$, and $\mathrm{Im}(\mathbf{z}) > 0$. Under these circumstances, the largest radius for a $\Lambda$-double lattice packing of disks is $\frac{1}{2}\min(1, \frac{|\mathbf{z}|\,|\mathbf{z}\pm 1|}{2\mathrm{Im}(\mathbf{z})})$ where the $\pm$ has the opposite sign from $\mathrm{Re}(\mathbf{z})$.*

*Proof.* By scaling and rotating, we may assume that $\Lambda$ is generated by 1 and some number $\mathbf{v}$ in the upper half-plane. From here, we resort to the action of the modular group $PSL(2, \mathbb{Z})$ on the upper half-plane. $PSL(2, \mathbb{Z})$ acts on the upper half-plane by linear fractional transformations. In the upper half-plane model of the hyperbolic plane, $PSL(2,\mathbb{Z})$ is the group of orientation preserving isometries. It is known that $PSL(2,\mathbb{Z})$ can be generated by a translation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and an inversion $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (or more formally, their equivalence classes). Given a basis $\{1, \mathbf{v}\}$ for $\Lambda$, the translation corresponds to changing the basis to $\{1, \mathbf{v}+1\}$ and the inversion corresponds to changing the basis to $\{1, -\frac{1}{\mathbf{v}}\}$ (which still has the same shape as $\Lambda$, although on a different scale). Thus, the action of $PSL(2,\mathbb{Z})$ can be used to choose a basis $\{1, \mathbf{z}\}$ where $\mathbf{z}$ is in a fundamental domain for the action. The standard choice for such a fundamental domain is $\{\mathbf{z} \in \mathbb{C} : |\mathrm{Re}(\mathbf{z})| \leq \frac{1}{2}, |\mathbf{z}| \geq 1, \text{ and } \mathrm{Im}(\mathbf{z}) > 0\}$.

Thus, we may assume that $\Lambda$ is generated by 1 and $\mathbf{z}$ where $|\mathrm{Re}(\mathbf{z})| \leq \frac{1}{2}$, $|\mathbf{z}| \geq 1$ and $\mathrm{Im}(\mathbf{z}) > 0$. One consequence of this is that 1 is the shortest element in $\Lambda$, so we may place nonoverlapping congruent disks at the points of $\Lambda$ as long the radius of the disks is no more than $\frac{1}{2}$. However, we also need to place disks at points $\mathbf{w} + \Lambda$, for some choice of $\mathbf{w} \in \mathbb{C} - \Lambda$. We must determine the optimal choice for $\mathbf{w}$ so as to maximize the disk radius. Of course, we may choose to place $\mathbf{w}$ in the parallelogram bounded by 1 and $\mathbf{z}$. The optimal location for $\mathbf{w}$ would be the point(s) which maximizes the distance to the nearest lattice point. Because of the restrictions on $\mathbf{z}$, the nearest lattice point will be one of the four vertices of the parallelogram bounded by 1 and $\mathbf{z}$. Finding the optimal location for $\mathbf{w}$ is then a simple geometry problem. We find that the optimal location for $\mathbf{w}$ is for it to be equidistant from at least three of the four vertices of the parallelogram.

If $\mathrm{Re}(\mathbf{z}) \geq 0$, then the three vertices $0, 1$, and $\mathbf{z}$ are a correct choice. One can quickly compute that $|\mathbf{w}| = |\mathbf{w} - 1| = |\mathbf{w} - \mathbf{z}| = \frac{|\mathbf{z}|\,|\mathbf{z}-1|}{2\mathrm{Im}(\mathbf{z})}$ and further that $|\mathbf{w} - (\mathbf{z}+1)| \geq |\mathbf{w}|$. This means the distance from a point in $\mathbf{w} + \Lambda$ to its nearest neighbor in $\Lambda$ is exactly $|\mathbf{w}|$. We already know that the shortest distance between two elements of $\Lambda$ (or two elements of $\mathbf{w} + \Lambda$) is 1. Thus, the shortest distance between two elements of $\Lambda \cup (\mathbf{w} + \Lambda)$ is $\min(1, |\mathbf{w}|) = \min(1, \frac{|\mathbf{z}|\,|\mathbf{z}-1|}{2\mathrm{Im}(\mathbf{z})})$.

If $\mathrm{Re}(\mathbf{z}) < 0$, reflect the lattice across the imaginary axis to obtain a lattice generated by 1 and $-\bar{\mathbf{z}}$. Then the shortest distance between two elements of $\Lambda \cup (\mathbf{w} + \Lambda)$ is $\min(1, \frac{|-\bar{\mathbf{z}}|\,|-\bar{\mathbf{z}}-1|}{2\mathrm{Im}(-\bar{\mathbf{z}})}) = \min(1, \frac{|\mathbf{z}|\,|\mathbf{z}+1|}{2\mathrm{Im}(\mathbf{z})})$ □

The preceding theorem computed the maximal radius for a $\Lambda$-double lattice packing of disks, but after having first performed scaling and rotations (some of which occurred during the various changes of lattice generators) that were not explicitly described. While rotation would not affect the disk radius, scaling would. In order to explicitly compute the disk radius in a fully general situation, we would need to keep track of the scalings we perform. To do this, we'd need to determine which elements of $PSL(2,\mathbb{Z})$ were used to take a given lattice to the type described in the preceding theorem.

**Definition 2.4.** *Given a lattice $\Lambda$ generated by 1 and $\mathbf{v}$, we let $r(\mathbf{v})$ be the maximal radius for nonoverlapping disks placed at the points of any $\Lambda$-double lattice (meaning that among all possible choices for $\mathbf{w}$, we choose the one which maximizes the disk radius).*

First though, it will be convenient to change our view of how the $PSL(2,\mathbb{Z})$ action tiles the upper half-plane. The standard fundamental domain for this action is not really the best choice for our purposes. Using that tile, there is no clear connection between the geometry of a particular tile and the manner in which $r(\mathbf{v})$ is evaluated within that tile. Thus, we switch to a tiling of the upper half-plane by ideal triangles.

Let $T(a,b,c)$ be the ideal triangle with vertices $a$, $b$, and $c$. The vertices are not included in $T(a,b,c)$, but the edges are. We will choose $T(0,1,\infty)$ as our primary point of reference. The action of $PSL(2,\mathbb{Z})$ tiles the upper half-plane with copies of $T(0,1,\infty)$, although some nontrivial elements of $PSL(2,\mathbb{Z})$ fix individual tiles.

**Lemma 2.5.** *Within $T(0,1,\infty)$, $r(\mathbf{v}) = \frac{1}{2}\min(1, |\mathbf{v}|, |\mathbf{v}-1|, \frac{|\mathbf{v}|\,|\mathbf{v}-1|}{2\mathrm{Im}(\mathbf{v})})$.*

*Proof.* Let $\Delta$ be the portion of $T(0,1,\infty)$ which lies within the standard $PSL(2,\mathbb{Z})$ domain. Then there are six distinct hyperbolically congruent (although sometimes reflected) copies of $\Delta$ lying within $T(0,1,\infty)$. See Figure 1.

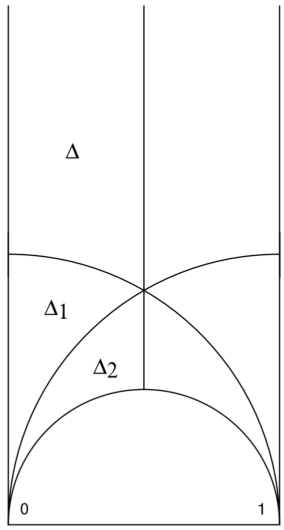


FIGURE 1.

We've already computed $r(\mathbf{v}) = \frac{1}{2}\min(1, \frac{|\mathbf{v}|\,|\mathbf{v}-1|}{2\mathrm{Im}(\mathbf{v})})$ within $\Delta$.

If we let $\Delta_1$ be the copy of $\Delta$ lying immediately below $\Delta$, then to compute $r(\mathbf{v})$ within $\Delta_1$, we must determine which element of $PSL(2,\mathbb{Z})$ carries $\Delta_1$ into the standard $PSL(2,\mathbb{Z})$ fundamental domain. Points in $\Delta_1$ have nonnegative real part, positive imaginary part, have length at most 1, and are at least 1 unit away from the point 1. If $\mathbf{v} \in \Delta_1$, then $-\frac{1}{\mathbf{v}}$ has, nonpositive real part, positive imaginary part, length at least 1, and real part at least $-\frac{1}{2}$. Such a point lies within the standard $PSL(2,\mathbb{Z})$ fundamental domain. However, the inversion changed the scale of the lattice by a factor of $|\mathbf{v}|$. Thus, $r(\mathbf{v}) = |\mathbf{v}|r(-\frac{1}{\mathbf{v}}) = \frac{|\mathbf{v}|}{2}\min(1, \frac{|-\frac{1}{\mathbf{v}}||-\frac{1}{\mathbf{v}}+1|}{2\mathrm{Im}(-\frac{1}{\mathbf{v}})}) = \frac{|\mathbf{v}|}{2}\min(1, \frac{\frac{1}{|\mathbf{v}|}\frac{|\mathbf{v}-1|}{|\mathbf{v}|}}{2\frac{\mathrm{Im}(\mathbf{v})}{|\mathbf{v}|^2}}) = \frac{1}{2}\min(|\mathbf{v}|, \frac{|\mathbf{v}||\mathbf{v}-1|}{2\mathrm{Im}(\mathbf{v})})$.

Letting $\Delta_2$ be the copy of $\Delta$ immediately below $\Delta_1$, we have that points in $\Delta_2$ have real part at most $\frac{1}{2}$, positive imaginary part, are at most 1 unit from 1, and are at least $\frac{1}{2}$ unit from $\frac{1}{2}$. For a point $\mathbf{v} \in \Delta_2$, the point $-\frac{1}{\mathbf{v}} + 1$ (inversion followed by translation) will have length at least 1, positive imaginary part, real part at most $\frac{1}{2}$, and nonnegative real part. Then $-\frac{1}{\mathbf{v}} + 1$ is in $\Delta$. However, the inversion changed the scale by $|\mathbf{v}|$, so $r(\mathbf{v}) = |\mathbf{v}|r(-\frac{1}{\mathbf{v}}+1) = \frac{|\mathbf{v}|}{2}\min(1, \frac{|-\frac{1}{\mathbf{v}}+1||-\frac{1}{\mathbf{v}}+1-1|}{2\mathrm{Im}(-\frac{1}{\mathbf{v}}+1)}) = \frac{|\mathbf{v}|}{2}\min(1, \frac{\frac{|\mathbf{v}-1|}{|\mathbf{v}|}\frac{1}{|\mathbf{v}|}}{2\frac{\mathrm{Im}(\mathbf{v})}{|\mathbf{v}|^2}}) = \frac{1}{2}\min(|\mathbf{v}|, \frac{|\mathbf{v}||\mathbf{v}-1|}{2\mathrm{Im}(\mathbf{v})})$.

Note that for $\mathbf{v} \in \Delta$, $|\mathbf{v}| \geq 1$ and $|\mathbf{v}-1| \geq 1$ so $r(\mathbf{v}) = \frac{1}{2}\min(1, \frac{|\mathbf{v}||\mathbf{v}-1|}{2\mathrm{Im}(\mathbf{v})}) = \frac{1}{2}\min(1, |\mathbf{v}|, |\mathbf{v}-1|, \frac{|\mathbf{v}||\mathbf{v}-1|}{2\mathrm{Im}(\mathbf{v})})$. Similarly, within either $\Delta_1$ or $\Delta_2$, $1 \geq |\mathbf{v}|$ and $|\mathbf{v}-1| \geq |\mathbf{v}|$ so $r(\mathbf{v}) = \frac{1}{2}\min(|\mathbf{v}|, \frac{|\mathbf{v}||\mathbf{v}-1|}{2\mathrm{Im}(\mathbf{v})}) = \frac{1}{2}\min(1, |\mathbf{v}|, |\mathbf{v}-1|, \frac{|\mathbf{v}||\mathbf{v}-1|}{2\mathrm{Im}(\mathbf{v})})$.

To compute $r(\mathbf{v})$ in the rest of $T(0,1,\infty)$, we note that the lattice generated by $\{1, \mathbf{v}\}$ has the same shape and scale as the lattice generated by $\{1, 1-\bar{\mathbf{v}}\}$, so $r(\mathbf{v}) = r(1-\bar{\mathbf{v}})$.

□

Knowing how $r(\mathbf{v})$ behaves within $T(0,1,\infty)$ is enough to allow us to compute it elsewhere. From here on, an ideal triangle $T(a,b,c)$ is assumed to be a $PSL(2,\mathbb{Z})$ translate of $T(0,1,\infty)$, rather than just being any arbitrary ideal triangle. This places some restrictions on $a$, $b$, and $c$, although the exact nature of the restrictions will not be relevant.

**Lemma 2.6.** *Within $T(a,b,\infty)$, $r(\mathbf{v}) = \frac{1}{2}\min(1, |\mathbf{v}-a|, |\mathbf{v}-b|, \frac{|\mathbf{v}-a||\mathbf{v}-b|}{2\mathrm{Im}(\mathbf{v})})$.*

*Proof.* This is easily verified, as any such ideal triangle, if translated far enough to the right or left becomes $T(0,1,\infty)$. □

**Lemma 2.7.** *If $a, b, c \in \mathbb{R}$, then within $T(a,b,c)$,*

$$r(\mathbf{v}) = \min(C_1|\mathbf{v}-a|, C_2|\mathbf{v}-b|, C_3|\mathbf{v}-c|, C_4\frac{|\mathbf{v}-a|\,|\mathbf{v}-b|\,|\mathbf{v}-c|}{\mathrm{Im}(\mathbf{v})})$$

*for some positive constants $C_1$, $C_2$, $C_3$, and $C_4$ which depend on only $a$, $b$, and $c$.*

*Proof.* Any ideal triangle under consideration can be carried to $T(0,1,\infty)$ by the action of $PSL(2,\mathbb{Z})$. That group is generated by inversions and translations, so it is sufficient to show three things:

(1) That the form of the indicated expression is invariant under translation.
(2) That the form of the indicated expression is invariant under inversion when none of $a$, $b$, and $c$ is 0.
(3) That when one of $a$, $b$, and $c$ is 0, inversion carries the indicated form to an expression of the type given in Lemma 2.6.

Translation by 1 carries $T(a,b,c)$ to $T(a+1,b+1,c+1)$ and performs no scaling. Suppose that $r(\mathbf{v}) = \min(C_1|\mathbf{v}-(a+1)|, C_2|\mathbf{v}-(b+1)|, C_3|\mathbf{v}-(c+1)|, C_4\frac{|\mathbf{v}-(a+1)|\,|\mathbf{v}-(b+1)|\,|\mathbf{v}-(c+1)|}{\mathrm{Im}(\mathbf{v})})$ within $T(a+1,b+1,c+1)$. Then within $T(a,b,c)$,

$$\begin{aligned}
r(\mathbf{v}) &= r(\mathbf{v}+1)\\
&= \min(C_1|\mathbf{v}+1-(a+1)|, C_2|\mathbf{v}+1-(b+1)|, C_3|\mathbf{v}+1-(c+1)|,\\
&\qquad C_4\frac{|\mathbf{v}+1-(a+1)|\,|\mathbf{v}+1-(b+1)|\,|\mathbf{v}+1-(c+1)|}{\mathrm{Im}(\mathbf{v}+1)})\\
&= \min(C_1|\mathbf{v}-a|, C_2|\mathbf{v}-b|, C_3|\mathbf{v}-c|, C_4\frac{|\mathbf{v}-a|\,|\mathbf{v}-b|\,|\mathbf{v}-c|}{\mathrm{Im}(\mathbf{v})}).
\end{aligned}$$

This verifies (1).

To verify (2), we assume that within $T(-\frac{1}{a},-\frac{1}{b},-\frac{1}{c})$, $r(\mathbf{v}) = \min(C_1|\mathbf{v}-\left(-\frac{1}{a}\right)|, C_2|\mathbf{v}-\left(-\frac{1}{b}\right)|, C_3|\mathbf{v}-\left(-\frac{1}{c}\right)|, C_4\frac{|\mathbf{v}-(-\frac{1}{a})|\,|\mathbf{v}-(-\frac{1}{b})|\,|\mathbf{v}-(-\frac{1}{c})|}{\mathrm{Im}(\mathbf{v})})$. Given a lattice generated by 1 and $\mathbf{v}$, inversion carries it to a lattice generated by 1 and $-\frac{1}{\mathbf{v}}$. The two lattices have the same shape, but the scale differs by a factor of $|\mathbf{v}|$. Then if $\mathbf{v}\in T(a,b,c)$,

$$\begin{aligned}
r(\mathbf{v}) &= |\mathbf{v}|r(-\frac{1}{\mathbf{v}})\\
&= |\mathbf{v}|\min(C_1\left|-\frac{1}{\mathbf{v}}+\frac{1}{a}\right|, C_2\left|-\frac{1}{\mathbf{v}}+\frac{1}{b}\right|, C_3\left|-\frac{1}{\mathbf{v}}+\frac{1}{c}\right|,\\
&\qquad C_4\frac{|-\frac{1}{\mathbf{v}}+\frac{1}{a}|\,|-\frac{1}{\mathbf{v}}+\frac{1}{b}|\,|-\frac{1}{\mathbf{v}}+\frac{1}{c}|}{\mathrm{Im}(-\frac{1}{\mathbf{v}})})\\
&= \min(C_1\left|\frac{\mathbf{v}}{a}-1\right|, C_2\left|\frac{\mathbf{v}}{b}-1\right|, C_3\left|\frac{\mathbf{v}}{c}-1\right|, C_4\frac{|\frac{\mathbf{v}}{a}-1|\,|\frac{\mathbf{v}}{b}-1|\,|\frac{\mathbf{v}}{c}-1|}{\mathrm{Im}(\mathbf{v})})\\
&= \min(\frac{C_1}{|a|}|\mathbf{v}-a|, \frac{C_2}{|b|}|\mathbf{v}-b|, \frac{C_3}{|c|}|\mathbf{v}-c|, \frac{C_4}{|abc|}\frac{|\mathbf{v}-a|\,|\mathbf{v}-b|\,|\mathbf{v}-c|}{\mathrm{Im}(\mathbf{v})}).
\end{aligned}$$

As this expression again has the desired form, we have verified (2).

Verifying (3) is similar to verifying (2). □

Now that we know the nature of $r(\mathbf{v})$, we can perform some computations to verify three technical lemmas.

**Lemma 2.8.** *Within a given ideal triangle, $r(\mathbf{v})$ is differentiable except on the curves along which there is a change in the identity of the minimal function.*

*Proof.* The function $r(\mathbf{v})$ is evaluated by finding the minimum of several functions, which are of the form $\frac{1}{2}$, $C|\mathbf{v}-a|$, $\frac{|\mathbf{v}-a|\,|\mathbf{v}-b|}{2\mathrm{Im}(\mathbf{v})}$, or $C\frac{|\mathbf{v}-a|\,|\mathbf{v}-b|\,|\mathbf{v}-c|}{\mathrm{Im}(\mathbf{v})}$. These functions are all obviously differentiable in the upper half-plane. Thus, the only possible points of nondifferentiability for $r(\mathbf{v})$ are points where two (or more) of the functions are simultaneously the minimum. Figure 2 indicates where within $T(0,1,\infty)$ each function is the minimum and where the points of nondifferentiability are. The curve on which $r(\mathbf{v})$ is nondifferentiable is indicated by the thick lines. $\square$

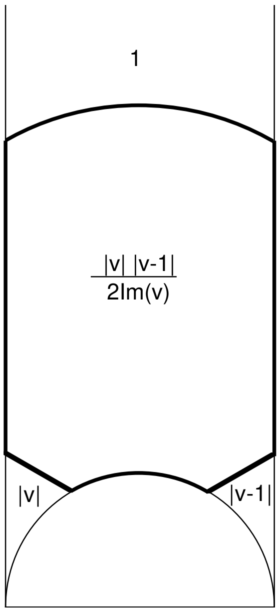


FIGURE 2.

**Lemma 2.9.** *$r(\mathbf{v})$ is an increasing function of* $\mathrm{Im}(\mathbf{v})$ *and is strictly increasing except when $r(\mathbf{v}) = \frac{1}{2}$.*

*Proof.* We need to check that $|\mathbf{v}-a|$, $\frac{|\mathbf{v}-a|\,|\mathbf{v}-b|}{\mathrm{Im}(\mathbf{v})}$, and $\frac{|\mathbf{v}-a|\,|\mathbf{v}-b|\,|\mathbf{v}-c|}{\mathrm{Im}(\mathbf{v})}$ are all strictly increasing functions of $\mathrm{Im}(\mathbf{v})$ in the appropriate portions of the upper half-plane. For $|\mathbf{v}-a|$, this is obvious.

The function $\frac{|\mathbf{v}-a|\,|\mathbf{v}-b|}{\mathrm{Im}(\mathbf{v})}$ is relevant within $T(a,b,\infty)$, for $a$ and $b$ consecutive integers. For notational convenience, let $\mathbf{v} = x + iy$. It would be sufficient to verify that $\frac{\partial}{\partial y}\left(\frac{((x-a)^2+y^2)((x-b)^2+y^2)}{y^2}\right) \geq 0$. The function is obviously not constant on any interval of $y$ values, so showing that the derivative is nonnegative is enough to guarantee strict monotonicity.

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{((x-a)^2+y^2)((x-b)^2+y^2)}{y^2}\right) &= -2\,\frac{(x-a)^2(x-b)^2}{y^3}+2y\\ &= 2\,\frac{y^4-(x-a)^2(x-b)^2}{y^3}\\ &= \frac{2}{y^3}\,(y^2-(x-a)(x-b))(y^2+(x-a)(x-b))\end{aligned}$$

Within the ideal triangle $T(a,b,\infty)$, $y>0$, $x$ is between $a$ and $b$, and $(x-\frac{a+b}{2})^2+y^2\geq\frac{1}{4}$. Then $\frac{2}{y^3}>0$ and $(x-a)(x-b)\leq 0$, so it's sufficient to check that $y^2+(x-a)(x-b)\geq 0$.

$$\begin{aligned}y^2+(x-a)(x-b) &\geq \frac{1}{4}-(x-\frac{a+b}{2})^2+(x-a)(x-b)\\ &= \frac{1}{4}-\frac{(a+b)^2}{4}+ab\\ &= \frac{1}{4}-\frac{(a-b)^2}{4}\\ &= 0.\end{aligned}$$

That completes the proof for the second type of function.

The third type of function $\frac{|\mathbf{v}-a|\,|\mathbf{v}-b|\,|\mathbf{v}-c|}{\mathrm{Im}(\mathbf{v})}$ is relevant within the ideal triangle $T(a,b,c)$. Without loss of generality, assume $a<b<c$. Within $T(a,b,c)$, $y>0$, $(x-\frac{a+c}{2})^2+y^2\leq\frac{(c-a)^2}{4}$, $(x-\frac{a+b}{2})^2+y^2\geq\frac{(b-a)^2}{4}$, and $(x-\frac{b+c}{2})^2+y^2\geq\frac{(c-b)^2}{4}$. Then $a\leq x\leq c$ and for $a\leq x\leq b$, $y^2\geq\frac{(b-a)^2}{4}-(x-\frac{a+b}{2})^2=-(x-a)(x-b)\geq 0$ while for $b\leq x\leq c$, $y^2\geq\frac{(c-b)^2}{4}-(x-\frac{b+c}{2})^2=-(x-b)(x-c)\geq 0$.

Again, it would be sufficient to verify that

$$\frac{\partial}{\partial y}\left(\frac{((x-a)^2+y^2)((x-b)^2+y^2)((x-c)^2+y^2)}{y^2}\right)\geq 0.$$

$$\begin{aligned}&\frac{\partial}{\partial y}\left(\frac{((x-a)^2+y^2)((x-b)^2+y^2)((x-c)^2+y^2)}{y^2}\right)\\ &= 4y^3+2y[(x-a)^2+(x-b)^2+(x-c)^2]-2\,\frac{(x-a)^2(x-b)^2(x-c)^2}{y^3}\\ &= \frac{2}{y^3}\,(2y^6+y^4[(x-a)^2+(x-b)^2+(x-c)^2]-(x-a)^2(x-b)^2(x-c)^2)\end{aligned}$$

Then for $a \le x \le b$,

$$\begin{aligned}
&\frac{\partial}{\partial y}\left(\frac{((x-a)^2+y^2)((x-b)^2+y^2)((x-c)^2+y^2)}{y^2}\right) \\
\ge\;& \frac{2}{y^3}\left(-2(x-a)^3(x-b)^3+(x-a)^2(x-b)^2[(x-a)^2+(x-b)^2+(x-c)^2]\right. \\
&\left.-(x-a)^2(x-b)^2(x-c)^2\right) \\
=\;& \frac{2}{y^3}(x-a)^2(x-b)^2(a-b)^2 \\
\ge\;& 0.
\end{aligned}$$

The computation for $b \le x \le c$ is nearly identical. □

**Lemma 2.10.** *On the level sets of the function $r(\mathbf{v}) = r(x+iy)$ the points with the lowest $y$ values are points of nondifferentiability.*

*Proof.* Recall that $r(\mathbf{v})$ is computed by determining the minimum of several functions. When exactly one of the functions is the minimum, $r(\mathbf{v})$ is differentiable. When two (or more) functions are simultaneously the minimum, $r(\mathbf{v})$ is not differentiable. This means that within any $T(a,b,c)$, there are four different regions in which $r(\mathbf{v})$ is differentiable. Where these regions meet each other, $r(\mathbf{v})$ is not differentiable. We can compute the level sets within each of the four zones of differentiability within each ideal triangle. The level sets of functions of the form $|\mathbf{v}-a|$ or $\frac{|\mathbf{v}-a|\,|\mathbf{v}-b|}{2\mathrm{Im}(\mathbf{v})}$ are circles, making it easy to locate the lowest point on the level set.

We have to do more work to verify the claim for a function of the form $\frac{|\mathbf{v}-a|\,|\mathbf{v}-b|\,|\mathbf{v}-c|}{\mathrm{Im}(\mathbf{v})}$. We may square the function without affecting the level sets, so let $f(\mathbf{v}) = \frac{|\mathbf{v}-a|^2\,|\mathbf{v}-b|^2\,|\mathbf{v}-c|^2}{(\mathrm{Im}(\mathbf{v}))^2}$. Let $\mathbf{v} = x+iy$. Then a level curve is a set of the form $f(x,y) = C$. We may regard $y$ as a function of $x$ along this curve. Differentiating with respect to $x$, $f_1 + f_2 y' = 0$. If the lowest point on the curve is a point of differentiability, then $y'$ would have to be 0 there. Differentiating again, $f_{11} + 2f_{12}y' + f_{22}(y')^2 + f_2 y'' = 0$. At the lowest point, $y' = 0$ and $y'' \ge 0$ which reduces the equation to $f_{11} + f_2 y'' = 0$. We know that $f_2$ is nonnegative so $f_{11}$ must be nonpositive. We shall show that this can't happen.

It would be enough to show that $y^2 f_{11} > 0$ in $T(a,b,c)$.

$$\begin{aligned}
y^2 f_{11} &= \frac{\partial^2}{\partial x^2}\left[((x-a)^2+y^2)((x-b)^2+y^2)((x-c)^2+y^2)\right] \\
&= 6y^4 + 4y^2\left[(x-a)^2+(x-b)^2+(x-c)^2+2(x-a)(x-b)\right. \\
&\quad \left.+2(x-b)(x-c)+2(x-a)(x-c)\right] + 2(x-a)^2(x-b)^2 \\
&\quad +2(x-b)^2(x-c)^2+2(x-a)^2(x-c)^2+8(x-a)(x-b)(x-c)^2 \\
&\quad +8(x-a)(x-b)^2(x-c)+8(x-a)^2(x-b)(x-c)
\end{aligned}$$

Note that the coefficient of $y^2$ simplifies to $4(3x-a-b-c)^2$, so is non-negative. As in the proof of the previous lemma, for $a \le x \le b < c$, we may assume that $y^2 \ge -(x-a)(x-b) \ge 0$. Then for $a \le x \le b$,

$$\begin{aligned}
y^2 f_{11} \;\ge\;& 6(x-a)^2(x-b)^2 - 4(x-a)(x-b)\big[(x-a)^2+(x-b)^2 \\
&+(x-c)^2+2(x-a)(x-b)+2(x-b)(x-c)+2(x-a)(x-c)\big] \\
&+2(x-a)^2(x-b)^2+2(x-b)^2(x-c)^2+2(x-a)^2(x-c)^2 \\
&+8(x-a)(x-b)(x-c)^2+8(x-a)(x-b)^2(x-c) \\
&+8(x-a)^2(x-b)(x-c) \\
=\;& -4(x-a)(x-b)\big[(x-a)^2+(x-b)^2\big] \\
&+2(x-b)^2(x-c)^2+2(x-a)^2(x-c)^2 \\
&+4(x-a)(x-b)(x-c)^2 \\
=\;& 8(a+b-2c)x^3+2(4c(2a+2b+c)-5(a+b)^2)x^2 \\
&+4(a+b)((a+b)^2-c(a+b+2c))x+2c^2(a+b)^2-4ab(a^2+b^2)
\end{aligned}$$

At this point, we have established that $y^2 f_{11}$ is bounded below by a cubic polynomial in $x$. The coefficient of $x^3$ is negative and the derivative of the polynomial is 0 at $x = \frac{a+b}{2}$ and $x = \frac{a+b+c}{3}$. In the interval $[a, b]$, the absolute minimum of the polynomial will occur either at the leftmost critical point, $\frac{a+b}{2}$ or at the right endpoint, $b$. Evaluating the polynomial at either of these points produces a positive number, so the polynomial is positive throughout the interval $[a, b]$. The polynomial is a lower bound for $y^2 f_{11}$, so $y^2 f_{11}$ is positive whenever $a \le x \le b$.

Repeating the same computations for $b \le x \le c$ completes the proof. □

These lemmas will allow us to restrict the possible maximally dense packings of disks of a given radius. Recall that when trying to locate optimal packings, we will require that there be a primitive element of length 1 in the lattice.

**Theorem 2.11.** *For a given disk radius $r'$ the highest density double lattice packing is of one of two types:*

1. *A lattice packing generated by two vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ of length $2r'$. There is an element of unit length of the form $m\mathbf{u_1} + n\mathbf{u_2}$ where either $m$ and $n$ are relatively prime odd integers or $\frac{m}{2}$ and $\frac{n}{2}$ are relatively prime integers one of which is even. Such a double lattice will be called an optimal lattice.*
2. *A double lattice $\Lambda \cup (\mathbf{w} + \Lambda)$ where $\Lambda$ is generated by a vector $\mathbf{u_1}$ of length $2r'$ and some other vector $\mathbf{u_2}$ of length at least $2r'$. The $\Lambda$-double lattice is a union of $\Lambda$ and $\mathbf{u_1}e^{i\pi/3} + \Lambda$. The point $\mathbf{u_1}e^{i\pi/3}$ is equidistant from 0 and the two generators of $\Lambda$. There is a primitive element of length 1 in $\Lambda$. Such a double lattice will be called an equilateral lattice.*

*Proof.* As usual, assume that $\Lambda$ is generated by 1 and some vector $\mathbf{v}$. We know that $r(\mathbf{v})$ is an increasing function of $\mathrm{Im}(\mathbf{v})$ and clearly $r(\mathbf{v})$ approaches 0 as $\mathrm{Im}(\mathbf{v})$ does. Choose some $\Lambda$-double lattice packing of disks of radius $r'$. We might as well assume that the disks are placed optimally, i.e. that even if $r' < r(\mathbf{v})$, the disks are centered at the same points as we could place disks of radius $r(\mathbf{v})$.

If $r' = \frac{1}{2}$, then the optimal double lattice packing is the hexagonal packing, which actually satisfies both aspects of the theorem. Thus, we may assume $r' < \frac{1}{2}$. If $r' < r(\mathbf{v})$, then by decreasing $\mathrm{Im}(\mathbf{v})$ slightly, we could still pack the disks of radius $r'$, and yet the area of the fundamental domain for $\Lambda$ (equal to $\mathrm{Im}(\mathbf{v})$) would decrease, thereby increasing density. Thus, we may assume that $r' = r(\mathbf{v})$.

At this point, we compute the density of the packing. Within a fundamental parallelogram for $\Lambda$ will lie portions of disks, totaling to two whole disks. The parallelogram has area $\mathrm{Im}(\mathbf{v})$, so the density is $\frac{2\pi(r')^2}{\mathrm{Im}(\mathbf{v})}$. Thus, for a given value of $r'$, the highest density will occur at the point on the level set $r' = r(\mathbf{v})$ with the smallest imaginary part. We have already shown that at such a point, $r(\mathbf{v})$ is nondifferentiable.

Returning to the region $\Delta$ used in the proof of Lemma 2.5, we see that the only points of nondifferentiability within $\Delta$ lie either on the boundary of $T(0,1,\infty)$ or on the curve $1 = \frac{|\mathbf{v}||\mathbf{v}-1|}{2\mathrm{Im}(\mathbf{v})}$. All other points of nondifferentiability anywhere in the upper half-plane will be $PSL(2,\mathbb{Z})$ translates of these (or their reflections about the imaginary axis), and thus the double lattices will still have the same shapes, although different sizes.

The only portion of the boundary of $T(0,1,\infty)$ that lies within $\Delta$ is on the line $\mathrm{Re}(\mathbf{v}) = 0$. It's easy to see that if $\mathrm{Re}(\mathbf{v}) = 0$ and $\mathrm{Im}(\mathbf{v}) > \sqrt{3}$ then $r(\mathbf{v})$ has the locally constant value $\frac{1}{2}$ so is in fact differentiable. Hence, we assume that $\mathrm{Im}(\mathbf{v}) \leq \sqrt{3}$. The lattice $\Lambda$ is a rectangular lattice. The second copy of the lattice is offset by $\mathbf{w} = \frac{1+\mathbf{v}}{2}$. Then $\Lambda \cup (\mathbf{w} + \Lambda)$ is also a lattice, which can be generated by $\frac{1+\mathbf{v}}{2}$ and $\frac{1-\mathbf{v}}{2}$. As $r(\mathbf{v}) = \frac{1}{2}\min(1, |\mathbf{v}|, |\mathbf{v}-1|, \frac{|\mathbf{v}||\mathbf{v}-1|}{2\mathrm{Im}(\mathbf{v})})$, the fact that $\mathbf{v}$ is pure imaginary and of restricted size indicates that $r(\mathbf{v}) = \frac{|1-\mathbf{v}|}{4} = \frac{|1+\mathbf{v}|}{4}$. Then $\frac{1\pm\mathbf{v}}{2}$ both have length $2r'$. This information can then be carried to the rest of the upper half-plane allowing us to say that the double lattice $\Lambda \cup (\mathbf{w} + \Lambda)$ is in general a lattice generated by two vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ of length $2r'$. Both $\mathbf{u_1}$ and $\mathbf{u_2}$ are in $\mathbf{w} + \Lambda$. The points in the double lattice will be of the form $m\mathbf{u_1} + n\mathbf{u_2}$ where $m$ and $n$ are integers. Among these points, the ones which fall into $\Lambda$ will be the ones for which $m$ and $n$ are either both odd or both even (since $\Lambda$ is generated by $\mathbf{u_1} + \mathbf{u_2}$ and $\mathbf{u_1} - \mathbf{u_2}$). The primitive (in $\Lambda$) element of unit length is then of the form $m\mathbf{u_1} + n\mathbf{u_2}$ where $m$ and $n$ are either both odd or both even. The fact that it's primitive in $\Lambda$ implies that if $m$ and $n$ have a common factor (other than 1), then dividing by the common factor must produce a vector which is not in $\Lambda$. Thus, the only possible common factor is 2 and in this case $\frac{m}{2}$ and $\frac{n}{2}$ can't both be odd. This provides the first type of optimal double lattice.

The second kind of nondifferentiability within $\Delta$ arises on the curve $1 = \frac{|\mathbf{v}||\mathbf{v}-1|}{2\mathrm{Im}(\mathbf{v})}$. For such values of $\mathbf{v}$, we find that $\mathbf{w} = e^{i\pi/3}$ and $r(\mathbf{v}) = \frac{1}{2}$. The points 0, 1, and $\mathbf{v}$ are all at a distance of 1 from $\mathbf{w}$. Carrying this information from $\Delta$ to the rest of the upper half-plane establishes the second type of optimal double lattice packing. □

## 3. Optimal Lattice Packings

Here, we will be studying the first of the two types of double lattice packings mentioned in Theorem 2.11, the optimal lattice packings.

**Theorem 3.1.** *For optimal lattice packings of disks of radius $r$, the maximal density is* $\max\limits_{|f(m,n,r)|\le\frac{1}{2}} \frac{\pi}{4\sqrt{1-f^2(m,n,r)}}$ *where* $f(m,n,r) = \frac{\frac{1}{4r^2}-2m^2-2n^2}{2(m^2-n^2)}$ *and $m$ and $n$ are relatively prime integers. There is an exceptional case when $r = \frac{1}{4}$. In this case, a hexagonal packing is possible, with density $\frac{\pi}{\sqrt{12}}$.*

*Proof.* Let the two generators of the lattice be $\mathbf{u}_1$ and $\mathbf{u}_2$, each of which has length $2r$. Let the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$ be $\theta$. Then the area of a fundamental parallelogram for the lattice is $4r^2\sin\theta$. Within such a parallelogram lie portions of disks totaling one whole disk, and thus of area $\pi r^2$. Then the density is $\frac{\pi}{4\sin\theta}$. To maximize density, we then need to minimize $\sin\theta$.

As the lattice provides a packing of nonoverlapping disks of radius $r$, there can't be any lattice elements shorter than $2r$. If $\sin\theta \le \frac{\sqrt{3}}{2}$, then $\mathbf{u}_1 - \mathbf{u}_2$ would be too short, so there is a lower bound on $\sin\theta$. We could stop here, but the result would then conclude only that density is at most $\frac{\pi}{\sqrt{12}}$, something we already knew.

There is an element $m\mathbf{u}_1 + n\mathbf{u}_2$ which has length 1. Computing the length of this vector, $4r^2(m^2+n^2+2mn\cos\theta) = 1$. Then (unless $mn = 0$), we have that $\cos\theta = \frac{\frac{1}{4r^2}-m^2-n^2}{2mn}$. Recall that either $m$ and $n$ are relatively prime odd integers or $\frac{m}{2}$ and $\frac{n}{2}$ are relatively prime integers, one of which is even. The restrictions on $m$ and $n$ can be expressed more simply through a change of variables $m = m'+n'$, $n = m'-n'$ where $m'$ and $n'$ are relatively prime. Then $\cos\theta = \frac{\frac{1}{4r^2}-2(m')^2-2(n')^2}{2((m')^2-(n')^2)} = f(m', n', r)$. We're trying to minimize $\sin\theta$ while maintaining $\sin\theta \ge \frac{\sqrt{3}}{2}$, which corresponds to maximizing $|f(m', n', r)| = |\cos\theta|$ while maintaining $|f(m', n', r)| \le \frac{1}{2}$.

Note that if $m$ or $n$ is large then $|\cos\theta| \approx \frac{m^2+n^2}{2|mn|} \ge 1 > \frac{1}{2}$, so finding the desired maximum is a finite problem.

In the exceptional case in which $n$ (respectively $m$) is 0, we have that $m\mathbf{u_1}$ is of length 1. Since $n$ is even, $m$ must also be even. Thus, $\frac{n}{2}$ and $\frac{m}{2}$ are relatively prime and of different parities. Since $\frac{n}{2}$ is still even, $\frac{m}{2}$ must be an odd number which is relatively prime to 0. Thus, $|m| = 2$ and $\mathbf{u_1}$ must have length $\frac{1}{2}$, so $r = \frac{1}{4}$. We quickly determine that a hexagonal packing is possible in this case. It is more natural to regard this case as an equilateral packing. □

We include a graph of the maximal density as a function of $\frac{1}{r}$ (Figure 3). Note that (aside from a lone point at $r = \frac{1}{4}$), there is a gap for $\frac{1}{\sqrt{28}} < r < \frac{1}{\sqrt{12}}$. Recall that optimal lattice packings arose when level curves for the function $r(\mathbf{v})$ intersected with the nondifferentiabilities of $r(\mathbf{v})$. For $\frac{1}{\sqrt{28}} < r < \frac{1}{\sqrt{12}}$, all such nondifferentiabilities occur at points corresponding to equilateral packings.

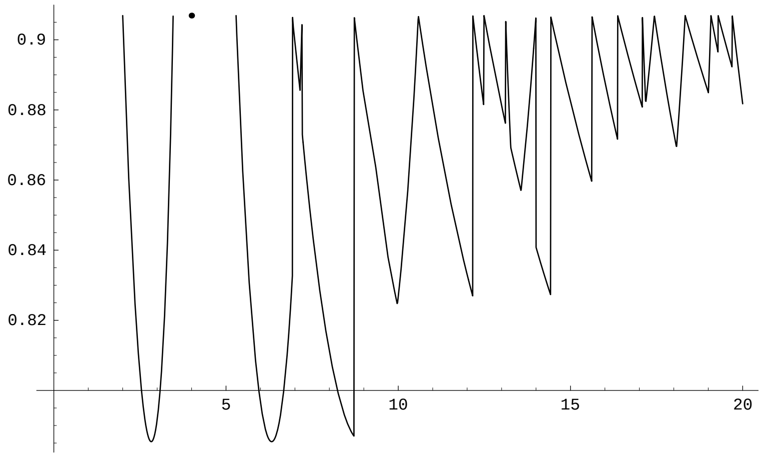


FIGURE 3.

## 4. EQUILATERAL PACKINGS

Here, we study the second type of packing mentioned in Theorem 2.11, the equilateral packings. We may rotate the double lattice so as to make $\mathbf{u}_1 = 2r$.

**Theorem 4.1.** *For equilateral packings of disks of radius $r$, the maximal density is* $\max_{g_\pm(m,n,r)\geq 2\sqrt{3}} \frac{\pi}{g_\pm(m,n,r)}$ *where* $\gcd(m,n) = 1$ *and*

$$g_\pm(m,n,r) = \frac{(m^2+mn+\frac{1}{4r^2})\sqrt{3}}{2(n^2+m^2+mn)} \pm (n+2m)\frac{\sqrt{\frac{n^2}{r^2} - (m(m+n) - \frac{1}{4r^2})^2}}{2(n^3+m^2n+mn^2)}.$$

*There is one exception. If $r = \frac{1}{2}$, a hexagonal packing is possible.*

*Proof.* The distance from $2re^{i\pi/3}$ to 0, $\mathbf{u}_1 = 2r$, and $\mathbf{u}_2$ is $2r$. Thus, the second generator is of the form $\mathbf{u}_2 = 2re^{i\pi/3} + 2re^{i\theta}$. If $\sin\theta < \frac{\sqrt{3}}{2}$, then one of $\mathbf{u}_2 \pm \mathbf{u_1}$ is within $2r$ of the point $2re^{i\pi/3}$, making a disk packing impossible. Thus we may assume that $\sin\theta \geq \frac{\sqrt{3}}{2}$.

We can readily compute that a fundamental parallelogram for $\Lambda$ has area $4r^2(\frac{\sqrt{3}}{2} + \sin\theta)$. This parallelogram contains portions of disks totaling two whole disks, and thus area $2\pi r^2$. Then the density is $\frac{\pi}{\sqrt{3}+2\sin\theta}$. To maximize density, we need to minimize $\sin\theta$. Using the lower bound we've already established, we could again prove that density is at most $\frac{\pi}{\sqrt{12}}$. However, we'd like to do better.

There is a primitive element $m\mathbf{u}_1 + n\mathbf{u}_2$ of length 1. Then $|2mr + nr + 2nr\cos\theta + inr\sqrt{3} + 2inr\sin\theta| = 1$ and $\gcd(m,n) = 1$. Determining the length of this vector, $m^2 + n^2(2+\cos\theta+\sqrt{3}\sin\theta) + mn(1+2\cos\theta) = \frac{1}{4r^2}$. Some algebra then verifies that (if $n \neq 0$)

$$\sin\theta = -\frac{(m^2+2n^2+mn-\frac{1}{4r^2})\sqrt{3}}{4(n^2+m^2+mn)} \pm (n+2m)\frac{\sqrt{\frac{n^2}{r^2} - (m(m+n)-\frac{1}{4r^2})^2}}{4(n^3+m^2n+mn^2)}.$$

We can now compute $\sqrt{3} + 2\sin\theta$, which is what's relevant for density.

$$\sqrt{3} + 2\sin\theta = \frac{(m^2+mn+\frac{1}{4r^2})\sqrt{3}}{2(n^2+m^2+mn)} \pm (n+2m)\frac{\sqrt{\frac{n^2}{r^2} - (m(m+n)-\frac{1}{4r^2})^2}}{2(n^3+m^2n+mn^2)}$$

Again, we note that finding the maximum is a finite problem, for if $m$ or $n$ is large and $\sin\theta \geq \frac{\sqrt{3}}{2}$, the equation involving the length of the vector becomes

$$\begin{aligned} \frac{1}{4r^2} &= m^2 + n^2(2+\cos\theta+\sqrt{3}\sin\theta) + mn(1+2\cos\theta) \\ &\geq \quad m^2 + 3n^2 - 2|mn| = \frac{m^2}{2} + n^2 + (\frac{|m|}{\sqrt{2}} - |n|\sqrt{2})^2 \gg \frac{1}{4r^2}. \end{aligned}$$

In the exceptional case $n = 0$, we have that $m\mathbf{u_1}$ is a primitive of length 1. Then $|m| = 1$ and $\mathbf{u}_1$ (which is of length $2r$) has length 1. Thus, $r = \frac{1}{2}$, and we get the hexagonal packing. It is more natural in this case to regard this as an optimal lattice packing, not an equilateral packing. $\square$

Again, we include a graph of the density as a function of $\frac{1}{r}$ (Figure 4). We point out that there is a gap in the graph for $\frac{1}{\sqrt{12}} < r < \frac{1}{2}$. The cause of this gap is similar to the cause of the gap in Figure 3.

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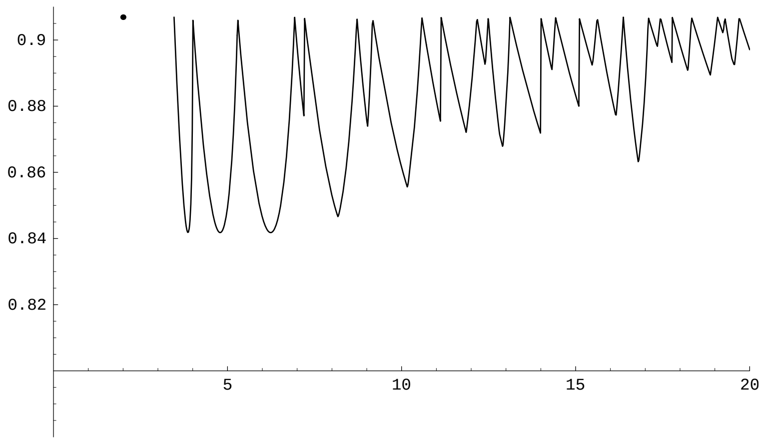


FIGURE 4.

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