DENSITY OF TUBE PACKINGS IN HYPERBOLIC SPACE

ANDREW PRZEWORSKI

Abstract. Given a hyperbolic manifold M and an embedded tube of radius r about some geodesic, we determine an upper bound on the percentage of the volume of M occupied by the tube.

1. Introduction

Packing problems have long been a topic of interest. Traditionally, efforts had been focused on Euclidean space, but as interest in hyperbolic space has grown, many of the Euclidean problems have been translated into the hyperbolic arena, in which the problems are almost always vastly more complicated.

The particular packing problem of interest here is a hyperbolic version of packing congruent right circular cylinders in Euclidean space. In Euclidean space, two equivalent ways to define a right circular cylinder are as the set of all points within a fixed distance of a given line or as the union of all lines passing perpendicularly through a given disk. In hyperbolic space, these two concepts are different. We will use the word *tube* in the former situation and the phrase *right circular cylinder* in the latter situation. Using this terminology, we are then investigating packings of congruent tubes in hyperbolic space.

Density is perhaps the primary focus in any investigation of packings. Unfortunately, density can be somewhat difficult to define in hyperbolic space, especially when one is dealing with objects of infinite volume. We will simplify the issue by dealing with only a certain class of packings, although the result would likely follow in more general settings, assuming one defined density properly.

Our main result is an upper bound on the density of symmetric packings of congruent tubes of radius r in hyperbolic space. We produce a means of computing the upper bound in arbitrary dimensions, and develop an explicit formula in dimension three. There is no reason to believe that our bounds are sharp, as we make a number of estimates along the way. We note that for the corresponding problem in three-dimensional Euclidean space, there is a sharp bound of $\frac{\pi}{\sqrt{12}}$ [BK90]. In $\mathbb{H}^3$, there is a prior result [MM00a], which provides an upper bound for very large radius tubes and is asymptotically sharp. The result we develop here works well for moderate radius tubes.

2000 *Mathematics Subject Classification.* Primary 51M09, 52C17, 57M50, 57N10.

However, for small radius tubes, our upper bound on density approaches 1, not the Euclidean upper bound $\frac{\pi}{\sqrt{12}}$ or the suspected hyperbolic limiting case of zero density. A further result [Prz02] deals with the small radius case. This later paper also includes an analysis of densities in a large number of known manifolds.

In section 7, we also produce various applications to the study of small volume hyperbolic three-manifolds.

2. Dirichlet domains for tube packings

Defining the density of a packing is often complicated. Since our applications for tube packings all concern tubes in finite volume manifolds, we will simply ignore the complications by dealing with only symmetric packings.

Definition 2.1. *A symmetric packing of tubes in $\mathbb{H}^n$ is a collection of nonoverlapping congruent tubes subject to the condition that the collection of tubes is preserved by the action of some discrete group $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ where $\mathbb{H}^n/\Gamma$ is a finite volume manifold. The density of the packing is the percentage of the volume of this manifold which is occupied by the projection of the tubes.*

Although our definition of density involves a manifold, we do not want to have to determine the manifold to determine density. For our purposes, it will be easier to deal with regions lying in hyperbolic space. Thus we consider one specific fundamental domain for the manifold.

Definition 2.2. *The Dirichlet domain of a tube T in a symmetric packing is the set of all points which are closer to the axis of T than to the axis of any other tube in the packing.*

As in the case of Dirichlet domains for sphere packings, the boundary of the Dirichlet domain will consist of $n-1$ dimensional manifolds (called faces) which are equidistant from two tubes. The point on a given face which lies on the common perpendicular to the two corresponding tubes is referred to as the center of the face. We note that some faces might not contain a center.

The Dirichlet domain of a tube will, of course, not be a finite volume object since it will contain the tube itself and the tube is of infinite length. Again, resorting to the symmetry, there is some action by translation along the tube, and this action will preserve the Dirichlet domain. This allows us to consider not the entire Dirichlet domain, but just some finite portion of it.

Definition 2.3. *A fundamental Dirichlet domain for a packing is a fundamental domain for the action of $\Gamma \cap \mathrm{Stab}(T)$ on a Dirichlet domain. We require that a fundamental Dirichlet domain have a limited type of convexity, specifically, any line segment perpendicular to the axis of T with one endpoint on the axis of T and the other on the boundary of the Dirichlet* *domain will either lie entirely within the fundamental Dirichlet domain or intersect it in at most a point on the axis of T.*

A fundamental Dirichlet domain, of course, will have the same volume as the quotient manifold $\mathbb{H}^n/\Gamma$.

Our approach to placing an upper bound on the density of the tube packing will be to place a lower bound on the volume of the region lying within the fundamental Dirichlet domain but outside the tube. We do this in two steps. First, we locate volume which lies near the center of a face and then we locate volume which lies far from the center of a face. In order to determine the density based on the effects of these two contributions, we will have to use a more localized concept of density.

Definition 2.4. *Let Ω be a finite area region lying on the boundary of a tube T. Let D be a Dirichlet domain for T. Consider the set $X_\Omega \subset D$ consisting of the union of all line segments which*

i) have one endpoint on the axis of T
ii) are perpendicular to the axis of T
iii) have one endpoint on ∂D and
iv) pass through Ω.

The density over the region Ω is defined to be the percentage of the volume of X_Ω which lies in T.

There is a simple relationship between the volume of $X_\Omega \cap T$ and Area(Ω), where Area is meant to be $n-1$ dimensional volume. We have three dimensional applications in mind, so will use terminology that is well-suited there. In $\mathbb{H}^3$ the relationship is $\mathrm{Vol}(X_\Omega \cap T) = \frac{1}{2}\tanh r \cdot \mathrm{Area}(\Omega)$.

If one takes the portion of ∂T which lies in a fundamental Dirichlet domain and divides it into various regions Ω_i then the density of the tube packing will be a weighted average of the densities over the Ω_i, with the weighting given by the areas of the Ω_i. In particular, we shall divide ∂T into regions corresponding to the faces of the Dirichlet domain and then subdivide each of those regions into points near the center of the face and points far from the center of the face. We will then establish upper bounds on the density over those regions. This will establish an upper bound on the density of the packing.

3. Cones in hyperbolic space

Our effort to develop an upper bound on density for tube packings will start by generalizing a result in [Prz01] which allows us to locate some volume that lies outside of the tubes. First, we define the region in question.

Definition 3.1. *Given two non overlapping tubes T_1 and T_2 of radius r, we take a ball B_i of radius r lying in T_i with center on the common perpendicular to the axes of T_1 and T_2. We define the region W to be the set of points which are closer to both B_1 and B_2 than to any other radius r ball which is disjoint from both B_1 and B_2.*

This construction parallels what was done in [Prz01]. As there, we see that the region W is a union of two right circular cones (when W is nonempty).

Proposition 3.2. *Let the distance between the axes of T_1 and T_2 be $2r+2d$. If $\tanh^2(r+d) < \tanh r \tanh 2r$, then W is nonempty and is the union of two right circular cones.*

Proof. Since the argument presented here is essentially identical to the one in [Prz01], we shall omit many of the details. Choose a point $p \in W$. Let B_3 be a radius r ball which is disjoint from B_1 and B_2. It is sufficient to consider the case in which B_3 is as close to p as possible. Note that since p lies in W, B_3 cannot contain p. We claim that the optimal position for B_3 is for it to be adjacent to both B_1 and B_2 with its center coplanar with p and the centers of B_1 and B_2. By taking a cross section in this plane, it is easy to complete the rest of the proof. □

Our interest is in tubes not balls, so we state a similar result involving tubes. From this point, we always assume that $\tanh^2(r+d) < \tanh r \tanh 2r$.

Proposition 3.3. *The points in the region W are closer to T_1 and T_2 than to any other radius r tube which is disjoint from T_1 and T_2.*

Proof. Choose a point $p \in W$. Let T_3 be a radius r tube which is disjoint from T_1 and T_2. Let B_1 and B_2 be as before and let B_3 be the radius r ball in T_3 which is closest to p. Since p is closer to B_1 and B_2 than to B_3, it is closer to T_1 and T_2 than to B_3. As the point in T_3 which is closest to p will lie on the boundary of B_3, we see that p is closer to T_1 and T_2 than to T_3. □

Finally, we consider the (nonempty) regions W_{ij} corresponding to all possible pairs of tubes T_i and T_j.

Proposition 3.4. *The interiors of the regions W_{ij} do not overlap each other.*

Proof. Choose a point p in W_{ij}. Determine the two tubes which are closest to p. These tubes must be T_i and T_j. This rules out the possibility that p also lies in W_{kl} where $\{i, j\} \neq \{k, l\}$. □

4. Points near face centers

The main result of [Prz01] can be used to determine a lower bound on the volume lying outside of a tube and near the center of a face which touches the boundary of the tube. We wish to generalize this to arbitrary faces and also modify it a little to make it easier to estimate density. We start by making some definitions.

Definition 4.1. *Given a face f of the Dirichlet domain for a tube T_1, construct the corresponding region W as in Definition 3.1 where T_2 is the tube on the opposite side of f. Let Σ be the intersection of ∂W with ∂B_1. The* *region Σ (if nonempty) will be an $n-2$ sphere (see Figure 1). Project Σ orthogonally onto the $n-1$ dimensional hyperplane Π passing through p_1 perpendicular to the altitude of the cones in W. This projection will also be an $n-2$ sphere. Let its radius be R.*

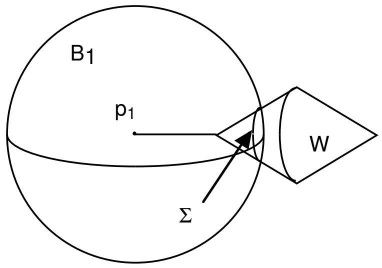

FIGURE 1

We now construct the region which we shall use to determine an upper bound on density.

Definition 4.2. *Let C_1 be the right circular cylinder whose*

i) base is the $n-1$ ball bounded by the projection of Σ onto Π.
ii) altitude lies on the (extended) altitude of W and is of length $r+d$.

Let C_2 be the corresponding cylinder constructed by exchanging the roles of T_1 and T_2.

We will show that the set $C = (C_1 \cup C_2) \setminus (T_1 \cup T_2)$ has the desired properties for a density computation. There are several things we need to verify.

Proposition 4.3. *$C \subset W$. As a result, the only Dirichlet domains that C intersects are the ones for T_1 and T_2.*

Proof. Because of the rotational symmetry of C_i about the altitude of W, it is sufficient to check this in a two dimensional cross section. We note that we need only verify that $C_1 \cap C \subset W$.

In two dimensions, we are dealing with the situation illustrated by Figure 2.

Here W reduces to a union of two isosceles triangles and C_1 is a quadrilateral with two right angles. Since the point at which ∂C_1 intersects W is at a distance of r from p_1 (by definition), we see that $C_1 \setminus T_1 \subset C \setminus B_1 \subset W$.

Because $C \subset W$, the points of C are closer to T_1 and T_2 than to any other tubes. Thus each point in C lies in either the Dirichlet domain for T_1 or the Dirichlet domain for T_2. □

At this point, we may partition the face f into regions near its center and regions far from its center. Let $\Omega = C_1 \cap \partial T_1$ and let X_Ω be the corresponding region as in Definition 2.4.

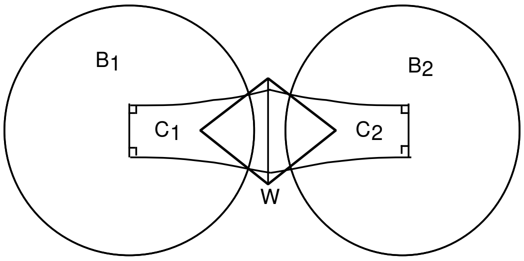

FIGURE 2

Proposition 4.4. $(C \cap D) \subset (X_\Omega \setminus T_1)$

Proof. If we extend C_1 to a semi-infinite cylinder C_1^∞, it will contain C_2. Further, we claim that $((C_1^\infty \cap D) \setminus T_1) \subset (X_\Omega \setminus T_1)$. To see this, take a cross section along any 2 dimensional hyperplane perpendicular to the axis of T_1. Such a cross section is indicated in Figure 3.

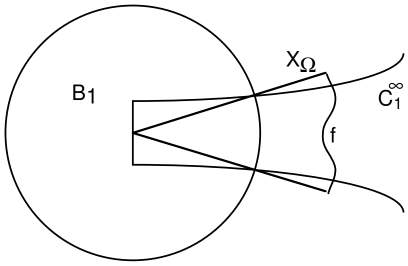

FIGURE 3

It is clear that within this cross section, $C \cap D \subset ((C_1^\infty \cap D) \setminus T_1) \subset (X_\Omega \setminus T_1)$ □

Of course, we could prove the same thing about T_2 and its Dirichlet domain. It is important to note the symmetry of $C, W, T_1 \cup T_2$ and f under the isometry which swaps T_1 and T_2, and thus that the portions of C lying in the two Dirichlet domains are congruent, so in particular have the same volume.

Proposition 4.5. $\mathrm{Vol}(X_\Omega \setminus T_1) \geq \mathrm{Vol}(C_1 \setminus T_1)$

Proof. Since $(C \cap D) \subset (X_\Omega \setminus T_1)$, $\mathrm{Vol}(X_\Omega \setminus T_1) \geq \mathrm{Vol}(C \cap D) = \frac{1}{2}\mathrm{Vol}(C) = \mathrm{Vol}(C_1 \setminus T_1)$. □

Proposition 4.6. *The density over the region* Ω *is at most*
$\left(1 + \frac{\mathrm{Vol}(C_1 \setminus T_1)}{\mathrm{Vol}(X_\Omega \cap T)}\right)^{-1}$

Proof.

$$\frac{\mathrm{Vol}(X_\Omega)}{\mathrm{Vol}(X_\Omega \cap T_1)} = 1 + \frac{\mathrm{Vol}(X_\Omega \setminus T_1)}{\mathrm{Vol}(X_\Omega \cap T_1)} \geq 1 + \frac{\mathrm{Vol}(C_1 \setminus T_1)}{\mathrm{Vol}(X_\Omega \cap T)}$$

The density over Ω is $\frac{\mathrm{Vol}(X_\Omega \cap T_1)}{\mathrm{Vol}(X_\Omega)}$, yielding the desired result. □

5. Points far from face centers

In the previous section, we determined an upper bound on the density contributed by points near face centers. We now need to deal with points which are not near face centers. First, we should be specific about which points are under consideration here.

Definition 5.1. *Given a face f, let Ω be defined as it was in the previous section. Let Ω^C be the set of points in $\partial T_1 \setminus \Omega$ through which we can produce a line segment which has one endpoint on the axis of T_1, has one endpoint in f, and is perpendicular to the axis of T_1. Denote the union of all such line segments X_{Ω^C}.*

As before, we will produce an upper bound on the density of $T_1 \cap X_{\Omega^C}$ within X_{Ω^C}. This will be achieved by placing a lower bound on $\mathrm{Vol}(X_{\Omega^C})$. Specifically, we shall determine a lower bound on the distance from the axis of T_1 to points in $f \cap X_{\Omega^C}$. By removing any part of X_{Ω^C} whose distance from the axis of T_1 is greater than this lower bound, we will have reduced X_{Ω^C} to its intersection with some tube which is coaxial with T_1 and of larger radius. It is then easy to compute the relevant volumes.

However, we'd prefer to avoid having to actually compute a distance function on f, so we take a somewhat less direct approach. We'll need to deal with the axes of the tubes here, so let l_i be the axis of T_i.

Proposition 5.2. *To determine a lower bound on $\inf_{p \in f \setminus C} \mathrm{dist}(p, l_1)$ it is sufficient to assume that l_1 and l_2 are coplanar. Let g_{min} denote the minimum distance in this situation.*

Proof. Let $g(p) = \max(\mathrm{dist}(p, l_1), \mathrm{dist}(p, l_2))$. For points p on the face f, $g(p) = \mathrm{dist}(p, l_1) = \mathrm{dist}(p, l_2)$. Then

$$\inf_{p \in f \setminus C} \mathrm{dist}(p, l_1) \geq \inf_{p \in \mathbb{H}^n \setminus (T_1 \cup T_2 \cup C)} g(p).$$

If we were to rotate l_1 and l_2 about their intersections with their common perpendicular then the value of $g(p)$ will be at least the value achieved when l_1, l_2 and p are coplanar. Thus, it is sufficient to consider p to be coplanar with l_1 and l_2. □

Proposition 5.3. *When l_1 and l_2 are coplanar, $\inf_{p \in \mathbb{H}^n \setminus (T_1 \cup T_2 \cup C)} g(p)$ occurs at a point on ∂C.*

Proof. It is sufficient to work within the plane containing l_1 and l_2. By moving p if necessary, we can reduce $g(p)$ unless p is equidistant from l_1 and l_2 or $p \in \partial C$. Within this two dimensional setting, the set of points equidistant from l_1 and l_2 is just a line midway between them. Along this line, $g(p)$ will decrease as p moves closer to the common perpendicular of l_1

and l_2. Hence, the exceptional case in which p is equidistant from l_1 and l_2 can be reduced to $p \in \partial C$. $\square$

Now, we relate this to a density estimate.

Proposition 5.4. *The density of D over Ω^C is at most the ratio of the volumes of tubes of radius r and g_{min}.*

Proof. The density of D over Ω^C is $\frac{\mathrm{Vol}(X_{\Omega^C} \cap T_1)}{\mathrm{Vol}(X_{\Omega^C})}$. Since X_{Ω^C} is a union of line segments all of length at least g_{min}, its volume is at least as great as the volume of the portion of X_{Ω^C} which lies within g_{min} of l_1. Since $X_{\Omega^C} \cap T_1$ is the portion of X_{Ω^C} which lies within r of l_1, the ratio of the two volumes is at most the ratio of the volumes of a tube of radius r with a tube of radius g_{min}. $\square$

6. Computations

An upper bound on density is described in the previous sections, but unless we can actually compute the upper bound, it is of little use. Here, we embark upon an effort to evaluate the many expressions involved. Some of the expressions are sufficiently complicated that we approximate them. The resulting upper bound on density is thus not as strong as possible.

As a start, we simply determine the value of R, the radius used in constructing the cylinder C_1. In order to do this, we will introduce some intermediate variables which we have not yet mentioned.

Let us introduce these variables as we recall how R was produced. Given two balls B_1 and B_2 of radius r whose centers p_1 and p_2 are separated by a distance $2r + 2d$, we situated a third radius r ball B_3 (center p_3) so as to have it tangent to each of the first two. Because of the rotational symmetry involved, we take a cross section along the plane containing the centers of the three balls. Consider the triangle $p_1p_2p_3$, and let γ be the angle $p_3p_2p_1$ (which is congruent to angle $p_3p_1p_2$). See Figure 4. Within this triangle, the cross section of the region W is the set of points lying closer to both p_1 and p_2 than to p_3. Of course, this region (if nonempty) will be bounded by the perpendicular bisectors of the segments p_1p_3 and p_2p_3. Along the bisector of p_2p_3 we locate the point q within W (if there is one) which is at a distance of r from p_1. Let β be the angle qp_1p_2. If we project q perpendicularly onto the line perpendicular to p_1p_2 through p_1 then R is the distance from the projection to p_1. If for any reason this construction fails, we set $R = 0$.

Proposition 6.1. *If* $\tanh r \tanh 2r \geq \tanh(r+d)\tanh(r+2d)$*, then β is determined by*

$$\begin{aligned} &\cosh r \cosh 2r - \cos(\gamma - \beta)\sinh r \sinh 2r \\ &= \cosh r \cosh(2r+2d) - \cos\beta \sinh r \sinh(2r+2d) \end{aligned}$$

and R is determined by $\tanh R = \tanh r \sin\beta$. *Otherwise, there is no point q so $R = 0$.*

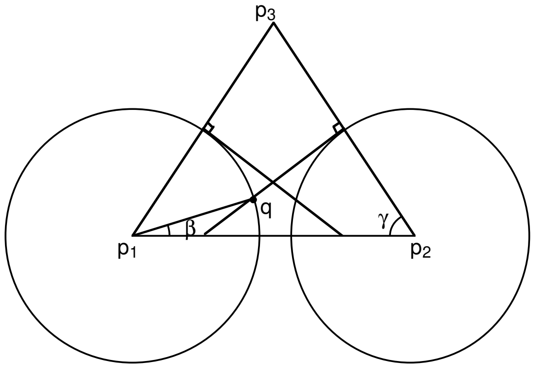

FIGURE 4

Proof. The point q is equidistant from p_3 and p_2. Using the law of cosines, we can determine the length of the segments qp_3 and qp_2. Equating these yields the desired expression.

As long as the perpendicular bisector of p_2p_3 intersects p_1p_2 at a point within r of p_1, there will be a point q. Constructing a right triangle using the bisector as one leg, half of p_2p_3 as the other and a portion of p_1p_2 as the hypotenuse, we find that the point q exists as long as the hypotenuse has length at least $r + 2d$. Using hyperbolic trigonometry, this requires that $\tanh r \cos\gamma \geq \tanh(r + 2d)$. We readily compute that $\cos\gamma = \frac{\tanh(r+d)}{\tanh 2r}$ and thus that q exists as long as $\tanh r \tanh 2r \geq \tanh(r + d)\tanh(r + 2d)$.

We then determine R by using hyperbolic trigonometry. ☐

(**Note:** In much of what follows, we shall assume that $R \neq 0$. The results are still true in the case in which $R = 0$, but they are often meaningless. When it matters, we will deal with the $R = 0$ case. Also, R is a function of r and d, although we will suppress that in the notation.)

We will occasionally need an upper bound on β.

Proposition 6.2. $\beta \leq \frac{\gamma}{2}$. *Equality is achieved only when* $d = 0$.

Proof. Since p_1p_2 is at least as long as p_1p_3, if the angle qp_1p_2 were larger than the angle qp_1p_3, it would follow that qp_2 would be longer than qp_3. Since qp_2 and qp_3 have the same length, we see that $\gamma - \beta \geq \beta$. The only case in which p_1p_2 and p_1p_3 have the same length is $d = 0$. ☐

Although this is not the order in which we worked earlier, it is quicker to determine g_{min} than to deal with the density over Ω.

Proposition 6.3. $\tanh g_{min} = \cosh R \tanh(r + d)$

Proof. It will be helpful in this argument to refer to Figure 5.

As was shown earlier, determining g_{min} reduces to a two dimensional computation. We have a pentagon with four right angles, with two unknown but equal sides forming the non-right angle at a vertex we shall call A. Across

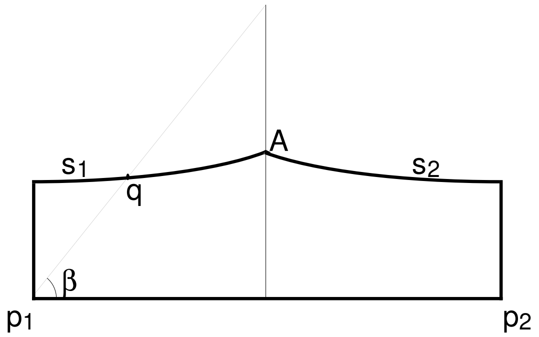

FIGURE 5

from this angle is the side p_1p_2 of length $2r+2d$. The remaining two sides have length R. We need to find the point(s) on the two unknown sides which minimize the function g, the larger of the distances to p_1 and p_2.

Given the nature of the function g, there are only two types of locations for the minimizing point(s). Either the point is equidistant from p_1 and p_2 or it is locally the closest point to either p_1 or p_2. We wish to eliminate the second possibility.

Clearly, on one of the sides, s_1, the right angled vertex minimizes the distance to p_1 and on the other s_2, the right angled vertex minimizes the distance to p_2. However, it should be equally clear that these points are not minima of g. If there is a point on s_1 which is a local minimum of the the distance to p_2, then the line joining this point to p_2 would form a right angle with s_1. That would force the (produced) angle p_1Ap_2 to be acute. We shall show that this can't happen.

The point q lies on s_1. Extend the line segment p_1q until it hits the (extended) bisector of the angle A. This produces a right triangle with β as one angle and the adjacent side of length $r+d$. The other angle α will be smaller than half of the angle p_1Ap_2. Thus it is sufficient to show that $\alpha \geq \frac{\pi}{4}$.

Form an isosceles triangle by adjoining another copy of this triangle along the leg opposite β. This triangle has base of length $2r+2d$, two angles of size β (at p_1 and p_2), and one angle of size 2α. By the law of cosines,

$$\begin{aligned}
\cos 2\alpha &= -\cos^2\beta + \sin^2\beta\cosh(2r+2d) \\
&= -1 + 2\sin^2\beta\cosh^2(r+d) \\
&\leq -1 + 2\sin^2\frac{\gamma}{2}\cosh^2(r+d) \\
&= -1 + (1-\cos\gamma)\cosh^2(r+d) \\
&= \sinh^2(r+d) - \frac{\sinh(r+d)\cosh(r+d)}{\tanh 2r} \\
&\leq \sinh^2(r+d) - \sinh(r+d)\cosh(r+d) < 0
\end{aligned}$$

Thus $\alpha \geq \frac{\pi}{4}$ so we have shown that g_{min} can be determined by considering points on s_1 which are equidistant from p_1 and p_2. Of course, the only such point is the vertex A. It is easy then to determine that $\tanh g_{min} = \cosh R \tanh(r+d)$. □

So far, we have not needed to know the dimension in which we are working. The arguments in the previous sections worked regardless of dimension and the computations have so far been independent of dimension. Unfortunately, the remaining computations involve volumes, which will, of course depend on the dimension. While we do not believe that it would be much more difficult to develop formulas which work in all dimensions, the computations are already fairly complicated in dimension three. Since we produce no applications of the result in higher dimensions, we restrict ourselves to dimension three from this point on.

The remaining part of our density estimate involves computing $\mathrm{Vol}(C_1 \setminus T_1)$ and $\mathrm{Area}(\Omega)$. It is not particularly difficult to determine these expressions, but they both end up being integrals which likely can't be evaluated in closed form. To simplify the computations, we shall approximate these expressions. We start with $\mathrm{Vol}(C_1 \setminus T_1)$.

Recall that C_1 is a right circular cylinder of radius R and height $r+d$ and that T_1 is the set of points which are within r of some specific line in the base of C_1.

Proposition 6.4.

$$\mathrm{Vol}(C_1 \setminus T_1) \geq \int_0^{2\pi} \int_0^R \int_r^{\tanh^{-1}(\cosh\rho \tanh(r+d))} \sinh\rho \cosh^2 z \, dz \, d\rho \, d\theta$$

Proof. We perform the computations in a cylindrical coordinate system (ρ, θ, z). Specifically, we choose a particular plane in $\mathbb{H}^3$ and establish a polar coordinate system (ρ, θ) on the plane. For an arbitrary point, z is the distance to the plane, and (ρ, θ) are the coordinates of the perpendicular projection of the point onto the plane. It is not too difficult to see that the volume element in this coordinate system is $\sinh\rho \cosh^2 z \, dz \, d\rho \, d\theta$.

We now take the $z = 0$ plane to be the base of C_1 and the line $\rho = 0$ to be the altitude of C_1. The "top" of C_1 is a plane parallel to $z = 0$ at a distance of $r + d$. We note that this is not the set $z = r + d$, which is not a plane. Rather, the top is the set $z = \tanh^{-1}(\cosh\rho \tanh(r + d))$ as is easy to verify. Finally, we need to compute the lower bound on z. Since points of $C_1 \setminus T_1$ are all at least r from some line in the $z = 0$ plane, using $z = r$ as a lower bound will only decrease the volume.

The bounds on ρ and θ should be obvious. □

We note that this integral can be evaluated in closed form.

Lastly, we must determine the area of Ω. Before we can do this, we'll have to find a parametrization for $\partial\Omega$, which will, of course, require a choice of a coordinate system. Since Ω lies on ∂T_1 which bears a natural Euclidean structure, we shall use that coordinate system. However, some of the intermediate computations will require coordinates on all of $\mathbb{H}^3$. We choose to work in the upper half space model.

Proposition 6.5. *In the natural Euclidean coordinates on ∂T_1, the boundary of Ω is the parametrized curve*

$$(\cosh r \ln \sqrt{\cosh 2R + \cos t \sinh 2R}, \sinh r \sin^{-1} \frac{\coth r \sin t \sinh R}{\sqrt{\cosh 2R + \cos t \sinh 2R}})$$

where $t \in [0, 2\pi]$.

Proof. In the upper half space, we shall place the axis of T_1 along the positive x_3 axis and place the base of C_1 in the plane $x_1 = 0$ with its center at $(0, 0, 1)$. It is then easy to see that the boundary of the base is the parametrized curve

$$(0, \sin t \sinh R, \cosh R + \cos t \sinh R)$$

for $t \in [0, 2\pi]$.

The "sides" of C_1 are surfaces consisting of line segments passing through this curve perpendicular to the base. In the upper half space model, these lines will be (Euclidean) circles. Because the circles are perpendicular to the $x_1 = 0$ plane (and the $x_3 = 0$ plane), they will be cross sections of (Euclidean) spheres centered at $(0, 0, 0)$. As a function of t, the radius of the sphere will be $\sqrt{\cosh 2R + \cos t \sinh 2R}$. Further, on a given circle, the x_2 coordinate will be fixed at $\sin t \sinh R$

We must determine where C_1 meets ∂T_1. In the upper half space model, ∂T_1 will be a (Euclidean) cone with vertex at the origin and vertex angle $\phi = \cos^{-1} \operatorname{sech} r$.

Thus, we must find the set of points which satisfy $x_2 = \sin t \sinh R$, are at a distance of $\sqrt{\cosh 2R + \cos t \sinh 2R}$ from $(0, 0, 0)$, and at an angle of ϕ from the x_3 axis. A simple trigonometric computation shows that the curve

$$(\sqrt{\sin^2 \phi(\cosh 2R + \cos t \sinh 2R) - \sin^2 t \sinh^2 R},$$
$$\sin t \sinh R, \cos \phi \sqrt{\cosh 2R + \cos t \sinh 2R}\ )$$

is the desired set. Actually, there would be a second copy with a negative x_1 value, but we have discarded that as C_1 exists on only one side of $x_1 = 0$. We have chosen that to be the positive side.

It is now easy to transfer to the Euclidean coordinates on ∂T_1 yielding the indicated curve. $\square$

Unfortunately, using this parametrization to compute the area of Ω would be complicated. We instead approximate the area with the area of a suitably sized ellipse.

Proposition 6.6. $\text{Area}(\Omega) \leq \pi R \cosh r \sinh r \sin^{-1} \frac{\tanh R}{\tanh r}$

Proof. First, we notice that performing a linear transformation on the coordinate system for ∂T will affect $\text{Area}(\Omega)$ only by scaling it. Thus, we scale by

$R\cosh r$ in the direction parallel to the axis of T_1 and by $\sinh r \sin^{-1} \frac{\tanh R}{\tanh r}$ in the perpendicular direction. The image of Ω is then bounded by

$$\left(\frac{1}{R}\ln\sqrt{\cosh 2R + \cos t \sinh 2R}, \frac{\sin^{-1}\frac{\coth r \sin t \sinh R}{\sqrt{\cosh 2R + \cos t \sinh 2R}}}{\sin^{-1}\frac{\tanh R}{\tanh r}}\right)$$

where $t \in [0, 2\pi]$.

Letting x be the first coordinate and y the second, we have that

$$\begin{aligned}
\sin^2 t &= \left[\frac{\tanh r}{\sinh R} e^{Rx} \sin(y \sin^{-1}\frac{\tanh R}{\tanh r})\right]^2 \\
1 - \cos^2 t &= \left[\frac{\tanh r}{\sinh R} e^{Rx} \sin(y \sin^{-1}\frac{\tanh R}{\tanh r})\right]^2 \\
1 - \left(\frac{e^{2Rx} - \cosh 2R}{\sinh 2R}\right)^2 &= \left[\frac{\tanh r}{\sinh R} e^{Rx} \sin(y \sin^{-1}\frac{\tanh R}{\tanh r})\right]^2 \\
2e^{2Rx}\cosh 2R - e^{4Rx} - 1 &= 4\left[\cosh R \tanh r e^{Rx} \sin(y \sin^{-1}\frac{\tanh R}{\tanh r})\right]^2 \\
\cosh 2R - \cosh 2Rx &= 2\left[\cosh R \tanh r \sin(y \sin^{-1}\frac{\tanh R}{\tanh r})\right]^2 \\
\sinh^2 R - \sinh^2 Rx &= \left[\cosh R \tanh r \sin(y \sin^{-1}\frac{\tanh R}{\tanh r})\right]^2 \\
1 - \frac{\sinh^2 Rx}{\sinh^2 R} &= \left[\frac{\sin(y \sin^{-1}\frac{\tanh R}{\tanh r})}{\frac{\tanh R}{\tanh r}}\right]^2
\end{aligned}$$

Thus, the transform of Ω is the region inside a curve of the form $\frac{\sinh^2 ax}{\sinh^2 a} + \frac{\sin^2 by}{\sin^2 b} = 1$. One can check that under certain circumstances, including $\sin b \geq \sinh a$ this curve bounds a region whose area is at most π. Thereafter, one need only check that $\frac{\tanh R}{\tanh r} \geq \sinh R$. This places the desired bound on Area(Ω). □

We are finally in a position to start making specific claims about tube density.

Proposition 6.7. *The density of a symmetric packing of tubes of radius r in $\mathbb{H}^3$ is at most the larger of*

$$\sup_d \left(1 + \frac{2\int_0^{2\pi}\int_0^R \int_r^{\tanh^{-1}(\cosh\rho \tanh(r+d))} \sinh\rho \cosh^2 z \, dz \, d\rho \, d\theta}{\pi R \sinh^2 r \sin^{-1}\frac{\tanh R}{\tanh r}}\right)^{-1}$$

and

$$\sup_d \frac{\sinh^2 r}{\sinh^2 \tanh^{-1}(\cosh R \tanh(r+d))}$$

Proof. The density of the tube packing is at most the larger of the density over Ω and the density over Ω^C. The latter of these should be fairly simple to compute, giving the second of the two functions in the statement of this proposition.

The density over Ω can be bounded above by the first function by incorporating the various results concerning the volume of $C_1 \setminus T_1$ and the area of Ω.

If d is large enough that $\tanh r \tanh 2r < \tanh(r+d)\tanh(r+2d)$, then $R = 0$ so Ω is empty, making the first expression irrelevant (and incomputable). The second expression simplifies to just $\frac{\sinh^2 r}{\sinh^2(r+d)}$. □

Proposition 6.8. *Both of the suprema in Proposition 6.7 are achieved when $d = 0$.*

Proof. This proof is a long and rather unpleasant computation. Presumably, one could also verify this statement numerically. Rather than reproduce the entire argument here, we shall indicate some of the key steps and leave the rest to the interested reader.

To start, we perform a change of variable $\rho = Ru$ in the triple integral, yielding:

$$\left(1 + \frac{2\int_0^{2\pi}\int_0^1\int_r^{\tanh^{-1}(\cosh Ru \tanh(r+d))} \sinh Ru \cosh^2 z \, dz \, du \, d\theta}{\pi \sinh^2 r \sin^{-1}\frac{\tanh R}{\tanh r}}\right)^{-1}$$

To establish that this is maximized when $d = 0$, it would be sufficient to show that

$$\frac{(\sinh Ru)\int_r^{\tanh^{-1}(\cosh Ru \tanh(r+d))} \cosh^2 z \, dz}{\sin^{-1}\frac{\tanh R}{\tanh r}}$$

is minimized when $d = 0$.

This can be evaluated rather easily to give, after rearrangement,

$$\left(\frac{\sinh Ru}{\sinh R}\right)\left(\frac{\frac{\tanh R}{\tanh r}}{\sin^{-1}\frac{\tanh R}{\tanh r}}\right)\tanh r \cdot \frac{\cosh R}{2}\left[\frac{\cosh Ru \tanh(r+d)}{1-\cosh^2 Ru \tanh^2(r+d)}\right.$$
$$\left. + \tanh^{-1}(\cosh Ru \tanh(r+d)) - \sinh r \cosh r - r\right]$$

After proving that R is a decreasing function of d, one sees that most of the factors in the above expression are easily dealt with, with the exception of $\cosh R$ and the bracketed expression. The negative terms in the bracketed expression can be ignored, leaving $\cosh R$ multiplied by a function of $v = \tanh^{-1}(\cosh Ru \tanh(r+d))$. We then factor $\sinh v$ out of the bracketed expression, yielding the product of $\cosh R \sinh v$ and an increasing function of v. Showing that $\cosh R \sinh v$ is an increasing function of d then shows that v is also an increasing function of d, finishing the proof.

To show that $\cosh R \sinh \tanh^{-1}(\cosh Ru \tanh(r+d))$ is increasing as a function of d, we first show that it's sufficient to assume that $u = 1$. With some fairly minimal computations, one then sees that it is sufficient to show that $\operatorname{sech}^2 R - \tanh^2(r+d)$ is a decreasing function of d. This computation is rather involved and difficult to describe succinctly, so we will stop here. □

Theorem 6.9. *The density of a symmetric packing of tubes of radius r in $\mathbb{H}^3$ is at most the larger of*

$$\left(1+\frac{2[\cosh R\tanh^{-1}(\cosh R\tanh r)-r-(\cosh R-1)(\frac{1}{2}\sinh 2r+r)]}{R\sinh^2 r\sin^{-1}\frac{\tanh R}{\tanh r}}\right)^{-1}$$

and

$$\frac{\sinh^2 r}{\sinh^2\tanh^{-1}(\cosh R\tanh r)}$$

where $\tanh R = \frac{\sinh r}{2\cosh^2 r}$. *Let $\rho(r)$ denote the value of the larger of these two functions.*

Proof. By substituting $d = 0$ in Proposition 6.7 and then evaluating the integral, we get the indicated expression. □

It appears to be the case that the former expression is always the larger, although we did not attempt to verify this, beyond plotting the two graphs. We also note that for large r, (roughly 7.1 or more), Marshall and Martin's asymptotic result [MM00a] is better than ours. Figure 6 is a graph of $\rho(r)$ for $r < 3$.

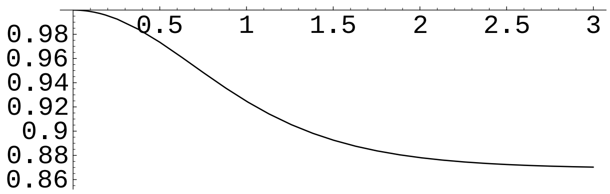

FIGURE 6

7. APPLICATIONS

There are various results concerning tubes in hyperbolic 3-manifolds and at the moment, Agol's [Ago02] is one of the strongest.

Theorem 7.1. [Ago02] *Let M be a hyperbolic 3-manifold and let γ be a geodesic link in M with an embedded open tubular neighborhood T of radius r. Let M_γ denote $M \setminus \gamma$ in a complete hyperbolic metric. Then*

$$\mathrm{Vol}(M_\gamma) \leq (\coth r\coth 2r)^{\frac{3}{2}}(\mathrm{Vol}(M) + (\frac{\coth r}{\coth 2r}-1)\mathrm{Vol}(T))$$

Agol proceeds by noting that $\mathrm{Vol}(T) \leq \mathrm{Vol}(M)$, thereby producing a relationship between r and the volumes of M and M_γ. We may now improve this estimate.

Corollary 7.2. *Let M be a hyperbolic 3-manifold and let γ be a geodesic link in M with an embedded open tubular neighborhood T of radius r. Let M_γ denote $M \setminus \gamma$ in a complete hyperbolic metric. Then*

$$\mathrm{Vol}(M) \geq (\tanh r \tanh 2r)^{\frac{3}{2}} \mathrm{Vol}(M_\gamma)(1 + \rho(r)(\frac{\coth r}{\coth 2r} - 1))^{-1}$$

Proof. $\mathrm{Vol}(T) \leq \rho(r)\mathrm{Vol}(M)$. Then one need only rearrange the terms. □

We now use this to improve estimates concerning small volume hyperbolic 3-manifolds.

Proposition 7.3. *All orientable hyperbolic 3-manifolds have volume at least* 0.324*.*

Proof. Cao and Meyerhoff have shown [CM01] that the minimal volume noncompact orientable hyperbolic 3-manifold has volume $2.0298\ldots$. The minimal volume orientable hyperbolic 3-manifold is known, by a result of Gabai, Meyerhoff, and Thurston [GMT03], to contain an embedded tube of radius at least $\frac{\log 3}{2}$ about its shortest geodesic. Using our improved version of Agol's result, we have that

$$\begin{aligned}
\mathrm{Vol}(M) &\geq \mathrm{Vol}(M_\gamma)(\tanh r \tanh 2r)^{\frac{3}{2}}(1 + \rho(r)(\frac{\coth r}{\coth 2r} - 1))^{-1} \\
&\geq 2.0298(\tanh \frac{\log 3}{2} \tanh \log 3)^{\frac{3}{2}}(1 + \rho(\frac{\log 3}{2})(\frac{\coth \frac{\log 3}{2}}{\coth \log 3} - 1))^{-1} \\
&\geq 0.324
\end{aligned}$$

□

Agol had already established a lower bound of 0.32, so our result represents only a very small improvement. This is in part because our density estimate is weaker for small tube radii. One can see a larger improvement in results concerning large tubes.

Proposition 7.4. *The shortest geodesic in the smallest volume orientable hyperbolic 3-manifold has length at least* 0.184 *and has an embedded tube about it of radius at most* 0.946*.*

Proof. Again, using our modified version of Agol's result, we can see that if $r > 0.946$ then $\mathrm{Vol}(M) \geq 0.943$, which is greater than the volume of the Weeks manifold. With this knowledge, we then resort to a result of Marshall and Martin [MM00b] which produces a lower bound on geodesic length, given tube radius. For tubes of radius between $\frac{\log 3}{2}$ and 0.946, we see that geodesic length is at least 0.184. □

The lower bound on geodesic length has been growing at a rapid pace, but as of now, the previous best known lower bound is 0.162 [HK02].

References

[Ago02] Ian Agol. Volume change under drilling. *Geom. Topol.*, 6:905–916 (electronic), 2002.

[BK90] A. Bezdek and W. Kuperberg. Maximum density space packing with congruent circular cylinders of infinite length. *Mathematika*, 37(1):74–80, 1990.

[CM01] Chun Cao and G. Robert Meyerhoff. The orientable cusped hyperbolic 3-manifolds of minimum volume. *Invent. Math.*, 146(3):451–478, 2001.

[GMT03] D. Gabai, R. Meyerhoff, and N. Thurston. Homotopy hyperbolic 3-manifolds are hyperbolic. *Ann. of Math. (2)*, 157(2):335–431, 2003.

[HK02] Craig D. Hodgson and Steven P. Kerckhoff. Universal bounds for hyperbolic dehn surgery, 2002. preprint.

[MM00a] T. Marshall and G. Martin. Cylinder and horoball packing in hyperbolic space, 2000. preprint.

[MM00b] T. Marshall and G. Martin. Volumes of hyperbolic 3-folds, 2000. to appear in J. Conf. Geom. and Dynamics.

[Prz01] Andrew Przeworski. Cones embedded in hyperbolic manifolds. *J. Differential Geom.*, 58(2):219–232, 2001.

[Prz02] Andrew Przeworski. A universal upper bound on density of tube packings in hyperbolic space, 2002. preprint.

Dept. of Mathematics, University of Texas, Austin, TX 78712

E-mail address: `prez@math.utexas.edu`

URL: `https://www.ma.utexas.edu/~prez/`