CONES EMBEDDED IN HYPERBOLIC MANIFOLDS

ANDREW PRZEWORSKI

Abstract. We show that the existence of a maximal embedded tube in a hyperbolic n-manifold implies the existence of a certain conical region. One application is to establish a lower bound on the volume of the region outside the tube, thereby improving estimates on volume and estimates on lengths of geodesics in small volume hyperbolic 3-manifolds. We also provide new bounds on the injectivity radius and diameter of an n-manifold.

1. Introduction

Lately, there has been much interest in tubes which embed in hyperbolic 3-manifolds. Among the results addressing either the consequences of or existence of such tubes are [Gab97],[GMT],[GMM],[GM98],[Mey87], and [Prz00]. Many of these results establish properties of the smallest volume hyperbolic orientable 3-manifold. The orientability condition is included since one of the main results [GMT] requires it to prove the existence of a tube of radius $\frac{\log 3}{2}$. Hence we shall take the definition of a manifold to include orientability. Specifically, Gabai, Meyerhoff, and Thurston [GMT] prove:

Theorem 1.1. [GMT] *Every closed orientable hyperbolic 3-manifold except Vol3 contains an embedded tube of radius at least* $0.52959\ldots$ *about its shortest geodesic. Unless the shortest geodesic has length greater than* $1.0595\ldots$*, there is a tube of radius* $\frac{\log 3}{2}$ *about this geodesic.*

We prove that if there is a maximal embedded tube of radius r then there is also another region W, defined later, which also embeds. This region is basically the union of two cones, whose shapes are determined by r. In addition, we prove that a certain portion of the region W lies outside of the tube of radius r.

In each of [GM98],[Prz00], and [GMM], lower bounds on tube volume were used to provide lower bounds on manifold volume. As we provide a lower bound on the volume outside the tube, these earlier results may be augmented. In particular, this author showed [Prz00] that any orientable hyperbolic 3-manifold has volume at least 0.276, which may now be improved to 0.28 and Gabai, Meyerhoff and Milley [GMM] showed that the smallest volume orientable hyperbolic 3-manifold contains no geodesic of length less than 0.069, which we may now improve to 0.09.

The techniques we use are not specific to 3-dimensions. Thus, we develop the result in arbitrary dimensions, even though most of our applications are specifically 3-dimensional. We do provide a few simple n-dimensional applications. Using the relative simplicity of the region W, we are able to determine the radius of the largest ball which fits inside W. This ball then embeds in the manifold. This allows us to place a lower bound on the radius of the largest ball embedded in a hyperbolic manifold.

Finally, we locate a point in W for which we can determine the distance to the geodesic. This then allows us to place a lower bound on the diameter of the manifold.

2. Establishing Precise Invariance

Let M^n be a hyperbolic n-manifold with fundamental group Γ. It is known that if M is closed then the shortest geodesic in M does not intersect itself and that there is thus an embedded tube about this geodesic. Even if M is not closed, there is often an embedded tube about some geodesic. Choosing some such geodesic, let r be the radius of the maximal embedded tube. We now change our viewpoint to that of the universal cover of M, $\mathbb{H}^n$. The group Γ can be considered as a subgroup of $\mathrm{Isom}^+(\mathbb{H}^n)$. The geodesic in M may be lifted to a line in $\mathbb{H}^n$ and the maximal tube may be lifted to the set of points within r of this line. The Γ action will mean that there are many choices of how to perform this lift. Let T_1 be some such lift of the maximal embedded tube and let T_2 be another such lift which intersects T_1 in a single point. We will denote the lines at the core of T_1 and T_2 as l_1 and l_2, respectively. As T_1 and T_2 intersect at a single point, l_1 and l_2 are at a distance of $2r$ from one another. Let q_1 and q_2 be the points at which their common perpendicular intersects l_1 and l_2, respectively. Let B_i be a ball of radius r about q_i. We construct a set which we shall show is precisely invariant under the action of Γ.

Definition 2.1. *Let X_1 be the union of all line segments having one endpoint at q_1 and the other endpoint in B_2. Let X_2 be the union of all line segments having one endpoint at q_2 and the other endpoint in B_1. Define the set $W = X_1 \cap X_2$.*

The set W will be two identical cones which have been attached along their bases. To be specific, we are defining a cone in $\mathbb{H}^n$ to be formed by an $n-1$ dimensional ball and a line segment, called the altitude, with one endpoint lying at the center of the ball such that the line segment is perpendicular to the $n-1$ dimensional hyperplane containing the ball. The set of points in the cone is the union of all line segments joining the other end of altitude to the ball. This is a higher dimensional version of a right circular cone.

We first prove a simple result based on this.

Lemma 2.2. *If $p \in W$ then* $\min_{i\in\{1,2\}} dist(p, q_i) < 2r$.

Proof. The points which are farthest from both q_1 and q_2 will be the points on the boundary of the base of the cones. Thus, it suffices to let p be one of these points. Then the ray $\vec{q_1 p}$ will intersect B_2 tangentially. Taking this ray, the segment $\overline{q_1 q_2}$, and the perpendicular dropped from q_2 to $\vec{q_1 p}$, we form a right triangle. The hypotenuse will be the segment $\overline{q_1 q_2}$ so will have length $2r$. The point p lies on one of the legs of the triangle. Thus $\text{dist}(p, q_1) < 2r$. □

We wish to show that W is precisely invariant under the Γ action. Most of the work lies in proving the following:

Proposition 2.3. *Let B_3 be a ball of radius r in $\mathbb{H}^n$ whose interior is disjoint from B_1 and B_2. Then for any $p \in W$, we have that dist$(p, B_3) \geq \max_{i \in \{1,2\}}$ dist(p, B_i). Equality holds only if $p \in \partial W$.*

Proof. Let q_3 be the center of B_3. By the preceding lemma, $p \neq q_3$ and, in fact, there is a positive lower bound on the possible distance from p to q_3. Without loss of generality, we may assume that q_3 is as close to p as allowed. Certainly, if B_3 intersects neither B_1 nor B_2, then it can be moved closer to p. So we may assume that B_3 is adjacent to at least one of B_1 and B_2.

Suppose for the moment that B_3 intersects B_1, but not B_2. Then, unless q_1, q_3 and p are colinear, we may move B_3 closer to p. So we take this as an additional assumption. If q_1 lies between p and q_3, then clearly p is closer to q_1 and q_2 than to q_3. Since p is within $2r$ of q_1, q_3 cannot lie between p and q_1. This leaves only the possibility that p lies between q_1 and q_3. However, as was shown in the proof of the preceding Lemma, there is a point of B_2 within $2r$ of q_1 along the ray $\vec{q_1 p}$. This point would be in the interior of B_3, a contradiction. The case in which B_3 intersects B_2 but not B_1 is dealt with similarly.

Hence, we may assume that B_3 intersects both B_1 and B_2. Since B_1, B_2, and W have a rotational symmetry about $\overline{q_1 q_2}$, there will be an $n-2$ sphere of possible locations for q_3. Along this sphere, the closest point to p will be one that lies in the plane Π containing q_1, q_2, and p. We have now reduced the situation to a two dimensional problem, illustrated by the following diagram.

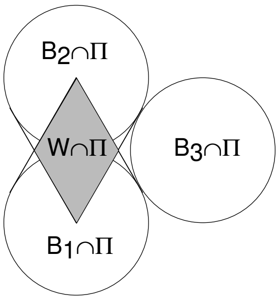

FIGURE 1

However, it is obvious in this situation that $\text{dist}(p, q_3) \geq \max_{i \in \{1,2\}} \text{dist}(p, q_i)$. It is also easy to see that equality holds only when p is equidistant from q_3 and either q_1 or q_2 and hence lies on ∂W. $\square$

We now prove a slightly stronger result.

Proposition 2.4. *Let T_3 be a tube of radius r in $\mathbb{H}^n$ whose interior is disjoint from T_1 and T_2. Then for any $p \in W$, we have that $dist(p, T_3) \geq \max_{i \in \{1,2\}} dist(p, T_i)$. Equality holds only if $p \in \partial W$*

Proof. Let l_3 be the axis of T_3 and let q_3 be the point of l_3 which is closest to p. Taking B_3 to be the ball of radius r about q_3, we have that $B_i \subset T_i$. The interior of B_3 will then be disjoint from B_1 and B_2. Since $\text{dist}(p, T_i) = \text{dist}(p, l_i) - r \leq \text{dist}(p, q_i) - r$, it would be sufficient to prove that $\text{dist}(p, l_3) = \text{dist}(p, q_3) \geq \max_{i \in \{1,2\}} \text{dist}(p, q_i)$. However, this is the result of the preceding proposition. $\square$

We are now ready to prove precise invariance.

Theorem 2.5. *The various translates of W under elements of Γ have no intersections, except at boundary points.*

Proof. A point p in the interior of W is closer to T_1 and T_2 than to any other Γ translates of T_1. If for some $\gamma \in \Gamma$, $\gamma(p) \in \text{int}\, W$ then $\gamma(p)$ is closer to T_1 and T_2 than to any other Γ translate of T_1. However, since γ is an isometry, $\gamma(p)$ is closer to $\gamma(T_1)$ and $\gamma(T_2)$ than to any other Γ translates of T_1. Hence, γ carries T_1 and T_2 to, in some order, T_1 and T_2. This implies that $T_1 \cap T_2$ is fixed by γ. The only element of Γ which has fixed points is the identity. $\square$

A simple consequence of this is:

Corollary 2.6. *The interior of W projects injectively to M. The portion of W lying outside $T_1 \cup T_2$ projects to a set which does not intersect the projection of T_1.*

Proof. That the interior of W projects injectively is obvious. To prove the remaining statement, let p be a point lying in $W \setminus (T_1 \cup T_2)$. If p were inside some Γ translate of T_1, it would be closer to that tube than to T_1 and T_2, which contradicts Proposition 2.4. $\square$

3. Hyperbolic Trig and Integration

Many of our applications will require complicated 3-dimensional computations. Rather than interrupt the flow of thought when we develop these applications, we will get the computations out of the way now, under the guise of computing the volumes of the 3-dimensional version of W and a specific region within W.

As was mentioned earlier, W is the union of two identical cones. Our first step will be to determine the exact shape of the cones. Let C denote the cone.

Proposition 3.1. *The altitude of C has length r. The vertex angle α is $\sin^{-1}\frac{\sinh r}{\sinh 2r}$ and the slant height is $\sinh^{-1}\frac{2\sinh r}{\sqrt{3}}$.*

Proof. It is obvious that the altitude of C is r. To determine the vertex angle, we consider that the side of the cone (when extended) will be tangent to a ball of radius r whose center is at a distance $2r$ from the vertex of C. This provides a right triangle whose hypotenuse is $2r$ and with one of the angles congruent to the vertex angle of C. The leg opposite the vertex angle is a radius of the ball so is of length r. The Law of Sines then determines the vertex angle. To determine the slant height, we consider a right triangle whose hypotenuse is the slant height. One angle is the vertex angle and the adjacent side has length r. Using the identity $\cos(\text{angle}) = \tanh(\text{adj})\coth(\text{hyp})$, we compute the slant height to be $\tanh^{-1}\frac{\tanh r}{\cos\sin^{-1}\frac{\sinh r}{\sinh 2r}} = \sinh^{-1}\frac{2\sinh r}{\sqrt{3}}$. $\square$

We note that he preceeding result does not depend on dimension. At this point we start to consider only the 3-dimensional case. We now derive a formula for the volume of a cone.

Proposition 3.2. *A right circular cone of altitude r and vertex angle α has volume $\pi\cos\alpha\tanh^{-1}\frac{\tanh r}{\cos\alpha} - \pi r$.*

Proof. We will represent the volume as a triple integral in spherical coordinates (ρ, ϕ, θ) where ρ is the distance to the origin, ϕ is the angle from some fixed line, and θ is an angle measured in a plane perpendicular to the fixed line. The necessary volume element is $\sin\phi\sinh^2\rho\, d\rho\, d\phi\, d\theta$.

We place the cone so the vertex lies at the origin and the altitude lies along the $\phi = 0$ direction. In an identical computation to the one performed in the preceding proof, we see that ρ varies between 0 and $\tanh^{-1}\frac{\tanh r}{\cos\phi}$. So the volume $V_C(r)$ is

$$\begin{aligned}
V_C(r) &= \int_0^{2\pi}\int_0^{\alpha}\int_0^{\tanh^{-1}\frac{\tanh r}{\cos\phi}} \sin\phi\sinh^2\rho\, d\rho\, d\phi\, d\theta \\
&= \pi\int_0^{\alpha}\int_0^{\tanh^{-1}\frac{\tanh r}{\cos\phi}} \sin\phi(\cosh 2\rho - 1)\, d\rho\, d\phi \\
&= \pi\int_0^{\alpha} \frac{\sin\phi\cos\phi\tanh r}{\cos^2\phi - \tanh^2 r} - \sin\phi\tanh^{-1}\frac{\tanh r}{\cos\phi}\, d\phi
\end{aligned}$$

At this point, we perform the substitution $u = \cos\phi$ and integrate the second half by parts.

$$\begin{aligned}
V_C(r) &= \pi\int_{\cos\alpha}^{1}\frac{u\tanh r\,du}{u^2-\tanh^2 r}-\pi\int_{\cos\alpha}^{1}\tanh^{-1}\frac{\tanh r}{u}\,du\\
&= \pi\int_{\cos\alpha}^{1}\frac{u\tanh r\,du}{u^2-\tanh^2 r}\\
&\quad -\pi\left(u\tanh^{-1}\frac{\tanh r}{u}\bigg|_{\cos\alpha}^{1}+\int_{\cos\alpha}^{1}\frac{u}{1-\frac{\tanh^2 r}{u^2}}\cdot\frac{\tanh r}{u^2}\,du\right)\\
&= \pi\cos\alpha\tanh^{-1}\frac{\tanh r}{\cos\alpha}-\pi r
\end{aligned}$$

□

We may now compute the volume of W.

Proposition 3.3. *The volume of W is*

$$2V_C(r)=\pi\sqrt{4-\operatorname{sech}^2 r}\,\sinh^{-1}\frac{2\sinh r}{\sqrt{3}}-2\pi r.$$

Proof. This is a simple consequence of plugging the relevant vertex angle and altitude into our cone volume formula and then simplifying. □

One of our main interests will be determining the volume of the region of W which lies in neither T_1 nor T_2. As the actual value is difficult to compute, we determine a lower bound. We do this by determining an upper bound on the volume of $W\cap T_1$. First, we give an intractable, but exact, formula for the volume.

Proposition 3.4. *The volume of $W\cap T_1$ is*

$$V(W\cap T_1)=\int_0^{2\pi}\int_0^{\alpha}\int_0^{\sinh^{-1}\frac{\sinh r}{\sqrt{1-\cos^2\theta\sin^2\phi}}}\sin\phi\sinh^2\rho\,d\rho\,d\phi\,d\theta.$$

Proof. Again we work in spherical coordinates. The axis of T_1 is perpendicular to the altitude of the cones in W. Placing the cone as before, we may take the axis of T_1 to lie in the $\phi=\frac{\pi}{2}$ plane in the direction of $\theta=0$ (and thus also $\theta=\pi$). The bounds on θ and ϕ are easy to establish. Also, the lower bound on ρ is clearly 0. To determine the upper bound, we will have to do a small amount of work. First we determine the angle β between the axis of T_1 and the line segment joining the origin to (ρ,ϕ,θ). Of course, ρ will not affect this angle. As is readily seen from the Poincaré disk model, we could just as well perform this computation in Euclidean space. As the spherical coordinates (ρ,ϕ,θ) correspond to the Cartesian coordinates $(\rho\cos\theta\sin\phi,\rho\sin\theta\sin\phi,\rho\cos\phi)$, the angle between this vector and the x-axis is given by $\cos\beta=\cos\theta\sin\phi$. Returning to hyperbolic geometry, we are now able to compute an upper bound on ρ. We need to know how far one can travel along a line at an angle of β from the axis of T_1, before one is at a distance of r from this axis. A quick application of the Law of Sines shows that $\sinh\rho\le\sinh r\csc\beta$. □

This expression is difficult to deal with so we make an approximation to simplify matters.

Proposition 3.5. *The volume of* $W \cap T_1$ *is less than*

$$V_T(r) = \pi \sinh r \left(\csc \tan^{-1} \frac{\cos\alpha}{\sinh r} - \csc \tan^{-1} \operatorname{csch} r\right) - \pi r + \pi \cos\alpha \sinh^{-1} \frac{\sinh r}{\cos\alpha}$$

Proof. Since $1 - \cos^2\theta \sin^2\phi \geq 1 - \sin^2\phi = \cos^2\phi$, we may say that

$$\begin{aligned}
V(W \cap T_1) &\leq \int_0^{2\pi} \int_0^{\alpha} \int_0^{\sinh^{-1} \frac{\sinh r}{\cos\phi}} \sin\phi \sinh^2\rho \, d\rho \, d\phi \, d\theta \\
&= \pi \int_0^{\alpha} \int_0^{\sinh^{-1} \frac{\sinh r}{\cos\phi}} \sin\phi (\cosh 2\rho - 1) \, d\rho \, d\phi \\
&= \pi \int_0^{\alpha} \sin\phi \left(\frac{\sinh r}{\cos\phi} \sqrt{1 + \frac{\sinh^2 r}{\cos^2\phi}} - \sinh^{-1} \frac{\sinh r}{\cos\phi} \right) d\phi
\end{aligned}$$

At this point, we make the substitution $u = \cos\phi$ and then integrate the second half by parts.

$$\begin{aligned}
V(W \cap T_1) &\leq \pi \int_{\cos\alpha}^{1} \frac{\sinh r}{u} \sqrt{1 + \frac{\sinh^2 r}{u^2}} \, du - \pi \int_{\cos\alpha}^{1} \sinh^{-1} \frac{\sinh r}{u} \, du \\
&= \pi \int_{\cos\alpha}^{1} \frac{\sinh r}{u} \sqrt{1 + \frac{\sinh^2 r}{u^2}} \, du - \\
&\quad \pi \left(u \sinh^{-1} \frac{\sinh r}{u} \Big|_{\cos\alpha}^{1} + \int_{\cos\alpha}^{1} u \cdot \frac{\frac{\sinh r}{u^2}}{\sqrt{1 + \frac{\sinh^2 r}{u^2}}} \, du \right) \\
&= \pi \int_{\cos\alpha}^{1} \frac{\sinh^3 r}{u^2 \sqrt{u^2 + \sinh^2 r}} \, du - \pi r + \pi \cos\alpha \sinh^{-1} \frac{\sinh r}{\cos\alpha}
\end{aligned}$$

Making the substitution $u = \sinh r \tan t$ allows us to complete the integration.

$$\begin{aligned}
V(W \cap T_1) &\leq \pi \int_{\tan^{-1} \frac{\cos\alpha}{\sinh r}}^{\tan^{-1} \operatorname{csch} r} \frac{\sinh r \cos t}{\sin^2 t} \, dt - \pi r + \pi \cos\alpha \sinh^{-1} \frac{\sinh r}{\cos\alpha} \\
&= \pi \sinh r \left(\csc \tan^{-1} \frac{\cos\alpha}{\sinh r} - \csc \tan^{-1} \operatorname{csch} r\right) - \pi r + \\
&\quad \pi \cos\alpha \sinh^{-1} \frac{\sinh r}{\cos\alpha}
\end{aligned}$$

□

This now allows us to compute a lower bound on the volume of the region of a manifold lying outside a tube of radius r.

Theorem 3.6. *Let M be an orientable hyperbolic 3-manifold containing an embedded tube of radius r about one of its geodesics. Then the region of M lying outside of this tube has volume at least $2(V_C(r) - V_T(r))$.*

Proof. In the case in which the embedded tube is of maximal radius, we have already shown that $W \setminus (T_1 \cup T_2)$ projects injectively to M and does not intersect the tube. Thus, we need only consider the case in which the tube is not of maximal radius. Expand the tube until it is of maximal radius R. Then again we know that there is a region of volume $2(V_C(R) - V_T(R))$ lying outside the maximal tube. All we need to note are that this region is, of course, outside of the original tube and that $2(V_C(r) - V_T(r))$ is an increasing function of r. $\square$

We perform one last calculation to determine the limiting behavior.

Proposition 3.7. $\lim_{r\to\infty} 2(V_C(r) - V_T(r)) = \pi(\log \frac{4}{3} - \frac{1}{4})$.

Proof.

$$\begin{aligned}
&\lim_{r\to\infty} 2(V_C(r) - V_T(r)) \\
&\quad = \lim_{r\to\infty} \pi\sqrt{4-\operatorname{sech}^2 r}\,\sinh^{-1}\frac{2\sinh r}{\sqrt{3}} \\
&\qquad -2\pi\sinh r\left(\frac{\sqrt{\sinh^2 r+\cos^2\alpha}}{\cos\alpha} - \cosh r\right) - 2\pi\cos\alpha\sinh^{-1}\frac{\sinh r}{\cos\alpha} \\
&\quad = \lim_{r\to\infty} \pi\sqrt{4-\operatorname{sech}^2 r}\,(r+\log\frac{2}{\sqrt{3}}) \\
&\qquad -2\pi\frac{\sinh r}{\cos\alpha}\left(\frac{\sinh^2 r+\cos^2\alpha-\cosh^2 r\cos^2\alpha}{\sqrt{\sinh^2 r+\cos^2\alpha}+\cosh r\cos\alpha}\right) - 2\pi\cos\alpha(r-\log\cos\alpha) \\
&\quad = 2\pi\log\frac{2}{\sqrt{3}} - 2\pi\lim_{r\to\infty}\frac{\sinh^2 r-\sinh^2 r\cos^2\alpha}{\sqrt{1+\frac{\cos^2\alpha}{\sinh^2 r}}+\coth r\cos\alpha} \\
&\quad = 2\pi\log\frac{2}{\sqrt{3}} - 2\pi\lim_{r\to\infty}\frac{\sinh^2 r\sin^2\alpha}{2} \\
&\quad = 2\pi\log\frac{2}{\sqrt{3}} - \frac{\pi}{4} = \pi(\log\frac{4}{3}-\frac{1}{4})
\end{aligned}$$

$\square$

4. Applications

We start by looking for embedded balls in hyperbolic manifolds.

Proposition 4.1. *If a hyperbolic manifold contains a tube of radius r about a geodesic then there is an embedded ball of radius $\sinh^{-1}(\frac{1}{2}\tanh r)$.*

Proof. Let r' be the radius of the maximal tube about this geodesic. The shortest distance from the center of W to the boundary will be achieved by dropping a perpendicular from the center to the boundary. This will form a right triangle with hypotenuse r', one angle equal to $\sin^{-1}\frac{\sinh r'}{\sinh 2r'}$ and the opposite leg the desired ball radius. It is easy to see that the radius of the ball is $\sinh^{-1}\frac{\sinh^2 r'}{\sinh 2r'} = \sinh^{-1}(\frac{1}{2}\tanh r')$, which is clearly increasing in r' and hence is at least $\sinh^{-1}(\frac{1}{2}\tanh r)$. □

This result allows us to prove two corollaries, one 3-dimensional and the other n-dimensional.

Corollary 4.2. *Every closed orientable hyperbolic 3-manifold contains a ball of radius at least* $\sinh^{-1}\frac{1}{4} = 0.24746\dots$.

Proof. If the manifold contains a tube of radius $\frac{\log 3}{2}$, then it must contain a ball of radius at least $\sinh^{-1}\frac{1}{4}$. Thus we need only note that the exceptional cases to the existence of the $\frac{\log 3}{2}$ tube also contain such a ball. □

For the n-dimensional result, we first cite a theorem of Cao and Waterman [CW98].

Theorem 4.3. *Let f be a geodesic in a complete hyperbolic n-manifold, with $k = \lfloor\frac{n-1}{2}\rfloor$ and $\sigma(p) = \int_0^{\frac{\pi}{2}} \sin^p x\,dx$. If the length l of f, is at most $\frac{2\sigma(k+1)}{\pi^k}\left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$, then there exists an embedded solid tube around f whose radius r satisfies*

$$\sinh^2 r = \frac{1}{4}\left(\frac{\pi^k l}{2\sigma(k+1)}\right)^{-\frac{2}{k+1}}\left(1-4\left(\frac{\pi^k l}{2\sigma(k+1)}\right)^{\frac{2}{k+1}}\right)^{\frac{1}{2}} - \frac{1}{2}.$$

Further, it is easy to see that r is a decreasing function of l. This gives:

Corollary 4.4. *If M^n is a complete hyperbolic n-manifold then either the shortest geodesic has length at least $\frac{2\sigma(k+1)}{\pi^k}\left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$ or there is an embedded ball of radius* 0.37. *As a consequence, every complete hyperbolic n-manifold contains a ball of radius $\frac{\sigma(k+1)}{\pi^k}\left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$.*

Proof. If the length l of the shortest geodesic is less than $\frac{2\sigma(k+1)}{\pi^k}\left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$ then there is a tube whose radius r satisfies $\sinh^2 r = \frac{1}{\sqrt{2}-1}\sqrt{2-\sqrt{2}} - \frac{1}{2} = 1.3477\dots$. This leads to a ball of radius at least $0.37\dots$.

If the shortest geodesic has length at least $\frac{2\sigma(k+1)}{\pi^k}\left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$ then there must be a ball of radius $\frac{\sigma(k+1)}{\pi^k}\left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$ about every point in M. Since this amount is smaller than 0.37 we get the desired lower bound on ball radius. □

We may state a corollary to this corollary:

Corollary 4.5. *If M^n is a complete hyperbolic n-manifold then Vol$(M) \geq V_n(1)\left(\frac{\sigma(k+1)}{\pi^k}\left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}\right)^n$ where $V_n(1)$ is the volume of an n-dimensional Euclidean ball of radius 1.*

Proof. We know that there is an embedded ball of radius $\frac{\sigma(k+1)}{\pi^k}\left(\frac{\sqrt{2}-1}{4}\right)^{\frac{k+1}{2}}$. The volume of a hyperbolic ball of a given radius is greater than the volume of the corresponding euclidean ball of the same radius. Thus we may use the Euclidean volume formula as a lower bound. $\square$

In addition to embedded balls, we can consider the diameter of a manifold.

Proposition 4.6. *If a hyperbolic manifold M contains an embedded tube of radius r about one of its geodesics then*

$$diam(M) \geq \sinh^{-1}\left(\frac{2\sinh r}{\sqrt{3}}\sqrt{1-\frac{\sinh^2 r}{\sinh^2 2r}}\right).$$

Proof. As we have shown, W embeds in M. We have also shown that the points of W are closer to T_1 or T_2 than any other $\pi_1(M)$ translate. Let us consider the points on the boundary of the bases of the cones in W. Among these points, the ones which are closest to l_1 will lie in the plane containing both l_1 and the common perpendicular to l_1 and l_2. Working within this plane, we have a point at a distance of $\sinh^{-1}\frac{2\sinh r}{\sqrt{3}}$ from q_1 and such that the angle of declination from l_1 is $\frac{\pi}{2}-\sin^{-1}\frac{\sinh r}{\sinh 2r}$. Thus, we may use the Law of Sines to see that the distance from this point to l_1 is $\sinh^{-1}\left(\frac{2\sinh r}{\sqrt{3}}\sqrt{1-\frac{\sinh^2 r}{\sinh^2 2r}}\right)$. The point could not be any closer to l_2 and thus l_1 is closer to the point than any other Γ translate. $\square$

From this point on, we restrict our attention to the 3-dimensional case. We first provide a very small improvement of an earlier result of this author [Prz00].

Proposition 4.7. *Every closed orientable hyperbolic 3-manifold has volume at least* 0.28.

Proof. The exceptional cases to the existence of a $\frac{\log 3}{2}$ tube all have volume at least 1.01. In [Prz00], it is shown that a tube of radius at least $\frac{\log 3}{2}$ in an orientable hyperbolic 3-manifold has volume at least 0.27666. Because of Theorem 3.6, we know that the region outside of the tube has volume at least 0.00485. $\square$

We now improve a recent result of Gabai, Meyerhoff, and Milley[GMM]. First, we state the relevant theorems from their paper:

Theorem 4.8. [GMM] *If a maximal tube in a complete orientable hyperbolic 3-manifold has length l and radius r and a value ρ is chosen less than $2r$ then l is at least as large as the smaller of*

$$\frac{\sqrt{3}\cosh(2r)}{2\pi\sinh(2r)}\left(\cosh^{-1}\left(\frac{\sinh^2(2r)+\cosh(2r+\rho)}{\cosh^2(2r)}\right)\right)^2$$

and

$$\frac{1}{\cosh(2r)\sinh(2r)}\left(\frac{{R_{2r,0}}^2}{4}+\left(R_{2r,\rho}-\frac{R_{2r,0}}{2}\right)^2\right)$$

where

$$R_{t,\rho}=\sqrt{\sinh(2r)\cosh(2r)\coth(2r+\rho)}\cosh^{-1}\left(\frac{\sinh(2r)\sinh(2r+\rho)+\cosh(t)}{\cosh(2r)\cosh(2r+\rho)}\right).$$

For their purposes, the value $\rho = 0.3$ was optimal. They then established various properties of these functions with $\rho = 0.3$. We will choose $\rho = 0.293$. Rather than reproduce their efforts, we say simply that with one exception, the exact same arguments would work for this new value of ρ. The exception is that the second function is decreasing so long as $r \geq 0.6$ whereas in [GMM] it is shown that this function is invertible whenever its value is less than 0.103.

Theorem 4.9. *The shortest geodesic in the smallest volume orientable hyperbolic 3-manifold has length at least* 0.09*.*

Proof. Suppose that there is a geodesic of length at most 0.09. In [GMM], it is shown that if a geodesic has length less than 0.10438 then there is a tube of radius at least 1.02 about it. Of course, this applies to our geodesic.

By multiplying the length estimates in Theorem 4.8 by $\pi\sinh^2 r$ we obtain lower bounds on tube volume. Thus, we can say that with $\rho = 0.293$, a tube of radius at least 1.332 has volume at least 0.87906. By Theorem 3.6, the volume outside this same tube is at least 0.06368. Thus the manifold has volume at least 0.94274 which is greater than the volume of the Weeks manifold. Thus, $1.02 \leq r \leq 1.332$.

Again using Theorem 4.8, we can say that $l \geq 0.09009$, a contradiction. ☐

It is also possible to improve estimates regarding noncompact manifolds. We would like to thank Peter Shalen for suggesting the following application.

Proposition 4.10. *If an orientable noncompact hyperbolic 3-manifold M has betti number at least 4 then Vol$(M) \geq \pi(\log\frac{4}{3}+\frac{3}{4})$*

Proof. In [ACCS96], it is established that there is a sequence of manifolds M_n such that $\pi_1(M_n)$ converges geometrically to $\pi_1(M)$. Further, $\mathrm{Vol}(M) > \mathrm{Vol}(M_n)$ and the limit as n goes to infinity of l_n, the length of the shortest geodesic in M_n is 0. It is also shown that M_n contains an embedded tube of volume at least $V(l_n)$ where V is a specific function they develop. All that we need to know about V is that $\lim_{l\to 0} V(l) = \pi$. As l_n goes to 0, the radius of the maximal embedded tube goes to ∞. This information is used to establish π as a lower bound on volume.

In M_n, by Theorem 3.6, we can establish a lower bound on the volume outside of the maximal tube. As $n \to \infty$, this lower bound will approach $\lim_{r\to\infty} 2(V_C(r) - V_T(r)) = \pi(\log \frac{4}{3} - \frac{1}{4})$. Thus, we may say that $\mathrm{Vol}(M) = \lim_{n\to\infty} \mathrm{Vol}(M_n) \geq \pi + \pi(\log \frac{4}{3} - \frac{1}{4}) = \pi(\log \frac{4}{3} + \frac{3}{4})$.

$\square$

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