AN UPPER BOUND ON DENSITY FOR PACKINGS OF COLLARS ABOUT HYPERPLANES IN $\mathbb{H}^n$

ANDREW PRZEWORSKI

Abstract. We consider packings of radius r collars about hyperplanes in $\mathbb{H}^n$. For such packings, we prove that the Delaunay cells are truncated ultra-ideal simplices which tile $\mathbb{H}^n$. If we place $n+1$ hyperplanes in $\mathbb{H}^n$ each at a distance of exactly $2r$ to the others, we could place radius r collars about these hyperplanes. The density of these collars within the corresponding Delaunay cell is an upper bound on density for all packings of radius r collars.

1. Introduction

In Euclidean geometry, an arrangement of disjoint codimension 1 hyperplanes must consist of parallel hyperplanes. If the hyperplanes were all at a distance of at least $2r$ from each other, we could inflate a collar of radius r about each hyperplane. These collars would have disjoint interiors. One can easily see that the densest packing of such collars is to have each collar immediately adjacent to two others, completely filling Euclidean space.

In hyperbolic geometry, it's actually impossible to fill space with disjoint collars around hyperplanes. Thus, the optimal density will be less than 1, and there is a packing problem to solve.

In an earlier paper [Prz10], we defined Delaunay cells for arrangements of flats in $\mathbb{H}^n$. Now, we prove that in the specific case of codimension 1 flats, the Delaunay cells are truncated ultra-ideal simplices and that they tile $\mathbb{H}^n$.

Theorem 3.5. *Let $\Pi_1, \cdots, \Pi_{m+1}$ in $\mathbb{H}^n$ be disjoint hyperplanes which are equidistant from some point p. If their closed Delaunay cell is m-dimensional, then it is a truncated ultra-ideal simplex. Each truncation face is contained in one of the Π_i. The truncation faces are perpendicular to all non-truncation faces (of any dimension) that intersect them.*

Theorem 3.8. *Let $\mathcal{P}$ be a symmetric cocompact arrangement of hyperplanes in $\mathbb{H}^n$ Then almost every point of $\mathbb{H}^n$ is in exactly one open n-dimensional Delaunay cell. In other words, the Delaunay cells tile $\mathbb{H}^n$.*

We will provide an upper bound on density for packings of collars about hyperplanes. This upper bound is achieved by placing $n+1$ hyperplanes in $\mathbb{H}^n$, each at a distance of exactly $2r$ from the others. We form a Delaunay

2010 *Mathematics Subject Classification.* Primary 51M09, 52C17.
Key words and phrases. density, hyperplanes, Delaunay, hyperbolic geometry.

cell which is a truncated regular ultra-ideal simplex and denote the density within this cell as $d_n(r)$.

Corollary 5.11. *For packings of radius r collars about hyperplanes in $\mathbb{H}^n$, the density is at most $d_n(r)$.*

This generalizes a two-dimensional result of Marshall and Martin [MM03].

In Section 2, we review the definition of Delaunay cells from [Prz10]. In Section 3, we prove Theorems 3.5 and 3.8. In Section 4, we use orthoschemes to study density and demonstrate that orthoschemes with shorter sides will achieve higher densities. Finally, in Section 5, we prove Corollary 5.11.

2. Definition of Delaunay cells

Here, we take some of the definitions and results from [Prz10] and condense them into the form needed for this paper. We provide no proofs here, but the proofs can be found in [Prz10]. When we need to use a model of $\mathbb{H}^n$, we use the Klein model, D^n.

Definition 2.1.

(1) If a hyperplane Π doesn't pass through the origin in D^n, then we represent Π by a vector $\mathbf{c} \in D^n$, the closest point on Π to $\mathbf{0}$. We will avoid considering hyperplanes that do pass through $\mathbf{0}$ and will often implicitly assume that hyperplanes don't pass through $\mathbf{0}$.
(2) The function $\pi : \mathbb{H}^n \to \Pi$ projects each point orthogonally onto Π. In the Klein model, $\pi(\mathbf{x}) = \mathbf{c} + \frac{1-|\mathbf{c}|^2}{1-\mathbf{x}\cdot\mathbf{c}}\left(\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{c}}{|\mathbf{c}|^2}\mathbf{c}\right)$

Definition 2.2. A point x in $\mathbb{R}^n$ or $\mathbb{H}^n$ is surrounded by the points $p_1, \cdots, p_{m+1}$ if $p_1, \cdots, p_{m+1}$ are in general position and x lies in the relative interior of their convex hull.

Lemma 2.3. *The point $\mathbf{x} \in \mathbb{R}^n$ is surrounded by the points $\mathbf{v}_1, \cdots, \mathbf{v}_{m+1}$ (using cyclic indices* $\bmod m+1$*) if and only if the multi-vectors $(-1)^{mj} \bigwedge_{i=1}^{m} (\mathbf{v}_{j+i} - \mathbf{x})$ for all values of j are nonzero and are positive scalar multiples of each other.*

Definition 2.4. Given pairwise disjoint hyperplanes $\Pi_1, \cdots, \Pi_{m+1} \in \mathbb{H}^n$ (for $1 \leq m \leq n$) and the corresponding hyperbolic projection functions $\pi_i : \mathbb{H}^n \to \Pi_i$, we define

(1) the open m-dimensional Delaunay cell associated to these hyperplanes to be $\{x \in \mathbb{H}^n \mid x \text{ is surrounded by the points } \pi_1(x), \cdots, \pi_{m+1}(x)\}$,
(2) the closed m-dimensional Delaunay cell associated with the hyperplanes to be the closure of the union of all open Delaunay cells associated with any subset of $\{\Pi_1, \cdots, \Pi_{m+1}\}$

When we say that two hyperplanes are disjoint, we mean that they have no intersection in $\mathbb{H}^n$ or in $\partial\mathbb{H}^n$. While Delaunay cells could still be defined in the event of hyperplanes which meet on $\partial\mathbb{H}^n$, many of the results we prove would require exceptional cases to handle such cells.

Lemma 2.5. *If $v \in \mathbb{H}^n$ is equidistant from the disjoint hyperplanes $\Pi_1, \cdots, \Pi_{m+1}$ (for $m \leq n$) then v lies in the component of $\mathbb{H}^n \setminus \bigcup_{i=1}^{m+1} \Pi_i$ which is bounded by all $m+1$ of the hyperplanes $\Pi_1, \cdots, \Pi_{m+1}$. The closed Delaunay cell lies within the closure of this component.*

In [Prz10], this Lemma was proved for $m = n$. The proof for $m < n$ is identical.

Definition 2.6. A symmetric cocompact arrangement of hyperplanes in $\mathbb{H}^n$ is a collection $\mathcal{P}$ of hyperplanes for which

(1) any two hyperplanes in $\mathcal{P}$ have no points in common within $\overline{\mathbb{H}^n}$,
(2) there is a discrete group $\Gamma < \mathrm{Isom}^+(\mathbb{H}^n)$ such that Γ acts transitively on $\mathcal{P}$ and $\mathbb{H}^n/\Gamma$ is a compact n-manifold.

Definition 2.7. Given a symmetric cocompact arrangement $\mathcal{P}$ of hyperplanes in $\mathbb{H}^n$, we can construct the Voronoi tessellation. Each Voronoi vertex is closest to (and equidistant from) $n + 1$ (or more) hyperplanes. For each vertex closest to exactly $n + 1$ of the hyperplanes, we form the closed Delaunay cell of those hyperplanes. For vertices closest to more than $n + 1$ of the hyperplanes, we follow a procedure for modifying the Voronoi tessellation (described in detail in Section 5 of [Prz10]) to produce multiple vertices, each associated with exactly $n + 1$ of the hyperplanes, and we then produce a Delaunay cell for each of those vertices.

Definition 2.8. We can speak of orientation within an n-dimensional Delaunay cell. Let T be the closed Delaunay cell associated with the hyperplanes $\Pi_1, \cdots, \Pi_{n+1}$, and let v be the Voronoi vertex equidistant from $\Pi_1, \cdots, \Pi_{n+1}$ (we assume that such a v exists, as we wouldn't otherwise have a reason to construct T). For any point $x \in T$, if the simplices $\pi_1(x) \cdots \pi_{n+1}(x)$ and $\pi_1(v) \cdots \pi_{n+1}(v)$ have the same/opposite orientation, we say that T has positive/negative (respectively) orientation at x.

For Delaunay cells associated with flats of arbitrary dimension in $\mathbb{H}^n$, it's possible to have negatively oriented Delaunay cells. It's also possible for multiple copies of the same Delaunay cell to be present. Thus, we can't expect for the Delaunay cells to tile $\mathbb{H}^n$. If we count the negatively oriented cells negatively and we count each occurrence of the repeated cells, we have the following result.

Theorem 2.9. *Let $\mathcal{P}$ be a symmetric cocompact arrangement of hyperplanes in $\mathbb{H}^n$. Counting with multiplicities and orientations, almost every point of $\mathbb{H}^n$ is in a total of one open n-dimensional Delaunay cell.*

In the following section, we will prove that for arrangements of hyperplanes, there are neither negatively oriented Delaunay cells nor repeated Delaunay cells.

3. Structure of Delaunay cells

In [Prz10], we proved that there is a map from a closed truncated simplex to the closed Delaunay cell, and that this map sends edges to one-dimensional Delaunay cells, 2-faces to two-dimensional Delaunay cells, etc. For arrangements of flats of arbitrary dimension, although the combinatorial structure of a Delaunay cell is always reasonably simple, the geometric structure is generally not. The faces are not usually totally geodesic, and it's possible for faces to have unexpected intersections with each other. However, for arrangements of hyperplanes, we will prove that even the geometric structure is nice.

Lemma 3.1. *Let Π_1 and Π_2 be disjoint hyperplanes in D^n (represented as $\mathbf{c}_1$ and $\mathbf{c}_2$ as in Definition 2.1), which are equidistant from the origin. Then the distance between Π_1 and Π_2 (in a hyperbolic sense) is* $\cosh^{-1} \frac{|\mathbf{c}_1|^2|\mathbf{c}_2|^2 - \mathbf{c}_1\cdot\mathbf{c}_2}{|\mathbf{c}_1||\mathbf{c}_2|\sqrt{1-|\mathbf{c}_1|^2}\sqrt{1-|\mathbf{c}_2|^2}}$.

Proof. The hyperbolic distance from the origin to the hyperplanes is $\tanh^{-1}|\mathbf{c}_i|$. We can form a pentagon from the three points $\mathbf{0}$, $\mathbf{c}_1$, $\mathbf{c}_2$, and the two points at which the common perpendicular to Π_1 and Π_2 meets Π_1 or Π_2. All of the angles in this pentagon will be right angles, except for the angle at $\mathbf{0}$, which is $\cos^{-1}\frac{\mathbf{c}_1\cdot\mathbf{c}_2}{|\mathbf{c}_1||\mathbf{c}_2|}$. The side opposite the angle at $\mathbf{0}$ is the common perpendicular to Π_1 and Π_2, so its length is the distance between the two planes.

From the trigonometry of hyperbolic pentagons (see, for example, [Fen89]), the cosh of the length of the common perpendicular is

$$\left(\sinh\tanh^{-1}|\mathbf{c}_1|\right)\left(\sinh\tanh^{-1}|\mathbf{c}_2|\right) - \left(\cosh\tanh^{-1}|\mathbf{c}_1|\right)\left(\cosh\tanh^{-1}|\mathbf{c}_2|\right)\frac{\mathbf{c}_1\cdot\mathbf{c}_2}{|\mathbf{c}_1||\mathbf{c}_2|}$$

After simplification, we see that the length of the common perpendicular is $\cosh^{-1}\frac{|\mathbf{c}_1|^2|\mathbf{c}_2|^2 - \mathbf{c}_1\cdot\mathbf{c}_2}{|\mathbf{c}_1||\mathbf{c}_2|\sqrt{1-|\mathbf{c}_1|^2}\sqrt{1-|\mathbf{c}_2|^2}}$. □

Note that if Π_1 and Π_2 intersect within $\mathbb{H}^n$, then the Lemma fails.

Lemma 3.2. *For a hyperplane Π represented as a vector $\mathbf{c} \in D^n$ (as in Definition 2.1),* $\pi(\mathbf{x}) - \mathbf{x} = \left(\frac{|\mathbf{c}|^2 - \mathbf{x}\cdot\mathbf{c}}{1-\mathbf{x}\cdot\mathbf{c}}\right)\left(\frac{\mathbf{c}}{|\mathbf{c}|^2} - \mathbf{x}\right)$

Proof. From Definition 2.1,

$$\begin{aligned}\pi(\mathbf{x}) - \mathbf{x} &= \mathbf{c} + \frac{1-|\mathbf{c}|^2}{1-\mathbf{x}\cdot\mathbf{c}}\left(\mathbf{x} - \frac{\mathbf{x}\cdot\mathbf{c}}{|\mathbf{c}|^2}\mathbf{c}\right) - \mathbf{x} \\ &= \left(\frac{|\mathbf{c}|^2 - \mathbf{x}\cdot\mathbf{c}}{(1-\mathbf{x}\cdot\mathbf{c})|\mathbf{c}|^2}\right)\mathbf{c} + \left(\frac{\mathbf{x}\cdot\mathbf{c} - |\mathbf{c}|^2}{1-\mathbf{x}\cdot\mathbf{c}}\right)\mathbf{x} \\ &= \left(\frac{|\mathbf{c}|^2 - \mathbf{x}\cdot\mathbf{c}}{1-\mathbf{x}\cdot\mathbf{c}}\right)\left(\frac{\mathbf{c}}{|\mathbf{c}|^2} - \mathbf{x}\right)\end{aligned}$$

□

Proposition 3.3. *For $m \leq n$, let $\Pi_1, \cdots, \Pi_{m+1}$ be disjoint hyperplanes (represented as vectors $\mathbf{c}_1, \cdots, \mathbf{c}_{m+1}$ as in Definition 2.1) which are equidistant from $\mathbf{0} \in D^n$. Let Δ be the relative interior of the convex hull of $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}, \cdots, \frac{\mathbf{c}_{m+1}}{|\mathbf{c}_{m+1}|^2}$ (which will include some points exterior to D^n).*

(1) *If $\mathbf{c}_1, \cdots, \mathbf{c}_{m+1}$ are in general position, then the open Delaunay cell associated with $\Pi_1, \cdots, \Pi_{m+1}$ is $\{\mathbf{x} \in \Delta \,|\, \mathbf{x}\cdot\mathbf{c}_i < |\mathbf{c}_i|^2 \text{for all } i \leq m+1\}$.*
(2) *If $\mathbf{c}_1, \cdots, \mathbf{c}_{m+1}$ aren't in general position, then the open Delaunay cell is empty.*

Proof. Each hyperplane Π_i partitions D^n into two components. The component containing the origin is $\{\mathbf{x} \in D^n \,|\, \mathbf{x}\cdot\mathbf{c}_i < |\mathbf{c}_i|^2\}$. By Lemma 2.5, the open Delaunay cell is in the same component of $D^n - \bigcup_{i=1}^{m+1} \Pi_i$ as the origin, so every point in the Delaunay cell satisfies $\mathbf{x}\cdot\mathbf{c}_i < |\mathbf{c}_i|^2$ for all $i \leq m+1$.

By Lemma 2.3, a point $\mathbf{x}$ is in the open Delaunay cell if and only if the multivectors $(-1)^{mj} \bigwedge_{i=1}^{m} (\pi_{j+i}(\mathbf{x}) - \mathbf{x})$ are nonzero for all values of j and are all positive scalar multiples of each other.

$$(-1)^{mj} \bigwedge_{i=1}^{m} (\pi_{j+i}(\mathbf{x}) - \mathbf{x}) = (-1)^{mj} \left(\prod_{i=1}^{m} \frac{|\mathbf{c}_{j+i}|^2 - \mathbf{x}\cdot\mathbf{c}_{j+i}}{1-\mathbf{x}\cdot\mathbf{c}_{j+i}}\right)\left(\bigwedge_{i=1}^{m}\left(\frac{\mathbf{c}_{j+i}}{|\mathbf{c}_{j+i}|^2} - \mathbf{x}\right)\right)$$

We need not consider any points for which $\mathbf{x}\cdot\mathbf{c}_{j+i} \geq |\mathbf{c}_{j+i}|^2$. Also, since $\mathbf{x}$ and $\mathbf{c}_{j+i}$ are both points in D^n, $1 - \mathbf{x}\cdot\mathbf{c}_{j+i} > 0$. With these restrictions, the condition that $(-1)^{mj} \bigwedge_{i=1}^{m} (\pi_{j+i}(\mathbf{x}) - \mathbf{x})$ are nonzero and are positive scalar multiples of each other is equivalent to the condition that $(-1)^{mj} \bigwedge_{i=1}^{m} \left(\frac{\mathbf{c}_{j+i}}{|\mathbf{c}_{j+i}|^2} - \mathbf{x}\right)$ are all nonzero and are positive scalar multiples of each other. Thus, $\mathbf{x}$ is surrounded by $\pi_1(\mathbf{x}), \cdots, \pi_{m+1}(\mathbf{x})$ if and only if $\mathbf{x}$ is surrounded by $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}, \cdots, \frac{\mathbf{c}_{m+1}}{|\mathbf{c}_{m+1}|^2}$. Rephrasing this statement, $\mathbf{x}$ is in the open Delaunay cell if and only if $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}, \cdots, \frac{\mathbf{c}_{m+1}}{|\mathbf{c}_{m+1}|^2}$ are in general position and $\mathbf{x}$ lies in the relative interior of their convex hull.

By assumption, all of the $\mathbf{c}_i$ have the same length. Then the vectors $\mathbf{c}_1, \cdots, \mathbf{c}_{m+1}$ are in general position if and only if the vectors $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}, \cdots, \frac{\mathbf{c}_{m+1}}{|\mathbf{c}_{m+1}|^2}$ are in general position. $\square$

Corollary 3.4. *Given disjoint hyperplanes $\Pi_1, \cdots, \Pi_{m+1}$ in $\mathbb{H}^n$ which are equidistant from some point p, their open Delaunay cell is either empty or m-dimensional.*

Theorem 3.5. *Let $\Pi_1, \cdots, \Pi_{m+1}$ in $\mathbb{H}^n$ be disjoint hyperplanes which are equidistant from some point p. If their closed Delaunay cell is m-dimensional, then it is a truncated ultra-ideal simplex. Each truncation face is contained in one of the Π_i. The truncation faces are perpendicular to all non-truncation faces (of any dimension) that intersect them.*

Proof. Without loss of generality, we may assume that p is placed at $\mathbf{0}$ in the Klein model. Represent hyperplane Π_i as a vector $\mathbf{c}_i \in D^n$, as in Definition 2.1. The closed Delaunay cell of $\Pi_1, \cdots, \Pi_{m+1}$ is formed by taking the union of all open Delaunay cells associated with any subset of $\{\Pi_1, \cdots, \Pi_{m+1}\}$ and then taking the closure of that object. Thus, if the closed Delaunay cell is m-dimensional, the open Delaunay cell associated with $\Pi_1, \cdots, \Pi_{m+1}$ must be m-dimensional, so $\mathbf{c}_1, \cdots, \mathbf{c}_{m+1}$ are in general position. The convex hull of $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}, \cdots, \frac{\mathbf{c}_{m+1}}{|\mathbf{c}_{m+1}|^2}$ is an m-dimensional simplex in $\mathbb{R}^n$, all of whose vertices lie outside of D^n. Thus, if we intersect this convex hull with D^n, we obtain an ultra-ideal simplex in hyperbolic space. Let $\overline{U}$ be the closure of the component of $D^n - \bigcup_{i=1}^{m+1} \Pi_i$ which contains $\mathbf{0}$. Let T be the intersection with $\overline{U}$ of the convex hull of $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}, \cdots, \frac{\mathbf{c}_{m+1}}{|\mathbf{c}_{m+1}|^2}$. Then T is a truncated ultra-ideal simplex. We claim that the closed Delaunay cell is T.

By repeated application of Proposition 3.3, the open Delaunay cell associated with any proper subset of $\{\Pi_1, \cdots, \Pi_{m+1}\}$ is the relative interior of a face of T, while the open Delaunay cell associated with $\{\Pi_1, \cdots, \Pi_{m+1}\}$ is the interior of T. All faces of T can be achieved in this fashion, except for the truncation faces, which lie in $\bigcup_{i=1}^{m+1} \Pi_i$, and thus aren't in any open Delaunay cell. Thus, the union of all open Delaunay cells associated with any subset of $\{\Pi_1, \cdots, \Pi_{m+1}\}$ is T with the truncation faces removed. The closed Delaunay cell is the closure of this object, so is T.

Let F be a truncation face of T. Without loss of generality, $F \subset \Pi_1$. The hyperplane Π_1 separates $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}$ from T. The hyperplane Π_1 also separates $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}$ from the other $\frac{\mathbf{c}_i}{|\mathbf{c}_i|^2}$ (since the hyperplanes are disjoint). Thus, if a face of the convex hull of $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}, \cdots, \frac{\mathbf{c}_{m+1}}{|\mathbf{c}_{m+1}|^2}$ intersects F, it must pass through the ultra-ideal vertex $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}$. Any flat which passes through $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}$ is perpendicular to Π_1 (in a hyperbolic sense). $\square$

Ushijima refers to such a polytope as a generalized simplex [Ush06].

Proposition 3.6. *Let $\Pi_1, \cdots, \Pi_{m+1}$ in $\mathbb{H}^n$ be disjoint hyperplanes which are equidistant from some point p. Let T be their closed Delaunay cell. If there is any point $x \in T \setminus \bigcup_{i=1}^{m+1} \Pi_i$ at which $\pi_1(x), \cdots, \pi_{m+1}(x)$ aren't in general position, then T has dimension less than m.*

Proof. Without loss of generality, $p = \mathbf{0} \in D^n$. Let $\mathbf{x} \in T \setminus \bigcup_{i=1}^{m+1} \Pi_i$ be such that $\pi_1(\mathbf{x}), \cdots, \pi_{m+1}(\mathbf{x})$ aren't in general position. Since $\mathbf{x}$ is in the open Delaunay cell associated with some subset of $\{\Pi_1, \cdots, \Pi_{m+1}\}$, $\mathbf{x}$ lies within the affine hull of $\pi_1(\mathbf{x}), \cdots, \pi_{m+1}(\mathbf{x})$, which has dimension less than m. By Lemma 3.2, $\pi_i(\mathbf{x}) - \mathbf{x} = \left(\frac{|\mathbf{c}_i|^2 - \mathbf{x}\cdot\mathbf{c}_i}{1-\mathbf{x}\cdot\mathbf{c}_i}\right)\left(\frac{\mathbf{c}_i}{|\mathbf{c}_i|^2} - \mathbf{x}\right)$. Since $\mathbf{x} \notin \bigcup_{i=1}^{m+1} \Pi_i$, there is no i for which $\frac{|\mathbf{c}_i|^2 - \mathbf{x}\cdot\mathbf{c}_i}{1-\mathbf{x}\cdot\mathbf{c}_i} = 0$. Then the points $\frac{\mathbf{c}_i}{|\mathbf{c}_i|^2}$ all lie within the affine hull of $\pi_1(\mathbf{x}), \cdots, \pi_{m+1}(\mathbf{x})$, so their affine hull is also of dimension less than m. Thus, the convex hull of $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}, \cdots, \frac{\mathbf{c}_{m+1}}{|\mathbf{c}_{m+1}|^2}$ has dimension less than m, so the open Delaunay cell associated with $\Pi_1, \cdots, \Pi_{m+1}$ is empty. The closed Delaunay cell is then the closure of the union of finitely many sets of dimension less than m, so is itself of dimension less than m. □

Proposition 3.7. *Let $\Pi_1, \cdots, \Pi_{n+1}$ in $\mathbb{H}^n$ be disjoint hyperplanes which are equidistant from some point p. Let T be their closed Delaunay cell. There are no points in $T \setminus \bigcup_{i=1}^{n+1} \Pi_i$ at which the cell has negative orientation.*

Proof. As usual, we place p at $\mathbf{0} \in D^n$ and represent each hyperplane Π_i as a vector $\mathbf{c}_i \in D^n$, as in Definition 2.1. All of the $\mathbf{c}_i$ are of the same length, which we denote $|\mathbf{c}|$.

By Lemma 3.1, the distance between Π_i and Π_j is $\cosh^{-1} \frac{|\mathbf{c}|^4 - \mathbf{c}_i\cdot\mathbf{c}_j}{|\mathbf{c}|^2(1-|\mathbf{c}|^2)}$, so $\frac{|\mathbf{c}|^4 - \mathbf{c}_i\cdot\mathbf{c}_j}{|\mathbf{c}|^2(1-|\mathbf{c}|^2)} \geq 1$ and thus $\mathbf{c}_i \cdot \mathbf{c}_j \leq 2|\mathbf{c}|^4 - |\mathbf{c}|^2 < |\mathbf{c}|^4$.

Assume that $\mathbf{c}_1, \cdots, \mathbf{c}_{n+1}$ aren't in general position. Then $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}, \cdots, \frac{\mathbf{c}_{n+1}}{|\mathbf{c}_{n+1}|^2}$ aren't in general position. For any $\mathbf{x} \in T$, the simplex $\pi_1(\mathbf{x}) \cdots \pi_{n+1}(\mathbf{x})$ lies within the affine hull of $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}, \cdots, \frac{\mathbf{c}_{n+1}}{|\mathbf{c}_{n+1}|^2}$. Thus, the simplex $\pi_1(\mathbf{x}) \cdots \pi_{n+1}(\mathbf{x})$ has volume 0, so the orientation of T isn't negative at $\mathbf{x}$.

Assume that $\mathbf{c}_1, \cdots, \mathbf{c}_{n+1}$ are in general position. Let $\mathbf{x} = \frac{\mathbf{c}_1 + \mathbf{c}_2}{2|\mathbf{c}|^2}$. The common perpendicular line to Π_1 and Π_2 must pass through both $\frac{\mathbf{c}_1}{|\mathbf{c}|^2}$ and $\frac{\mathbf{c}_2}{|\mathbf{c}|^2}$, so $\mathbf{x}$ is a point on the common perpendicular. Further, since $\mathbf{0}$ lies between Π_1 and Π_2, $\mathbf{x}$ does as well. Then $\mathbf{x}$ is a point in the open Delaunay cell associated with Π_1 and Π_2, so is a point in T. For any $i > 2$, $\mathbf{x} \cdot \mathbf{c}_i = \frac{1}{2|\mathbf{c}|^2}(\mathbf{c}_1 + \mathbf{c}_2) \cdot \mathbf{c}_i < |\mathbf{c}|^2$, so $\mathbf{x}$ can't be a point on any of the Π_i. Then $\mathbf{x} \in T \setminus \bigcup_{i=1}^{n+1} \Pi_i$. Note that $\mathbf{x} \cdot \mathbf{c}_1 = \mathbf{x} \cdot \mathbf{c}_2$.

The signed volume of the simplex $\pi_1(\mathbf{x}), \cdots, \pi_{n+1}(\mathbf{x})$ will vary continuously within T, and by Proposition 3.6, is never zero within the connected set $T \setminus \bigcup_{i=1}^{n+1} \Pi_i$. Thus, if we can prove that the orientation is positive at $\mathbf{x}$, we'll have proved that the orientation is positive everywhere within $T \setminus \bigcup_{i=1}^{n+1} \Pi_i$.

To determine the orientation at $\mathbf{x}$, we need to compare the orientation of the simplex $\pi_1(\mathbf{x}) \cdots \pi_{n+1}(\mathbf{x})$ to the orientation of the simplex $\mathbf{c}_1 \cdots \mathbf{c}_{n+1}$. We will do this by considering the exterior product $\bigwedge_{i=2}^{n+1} (\pi_i(\mathbf{x}) - \pi_1(\mathbf{x}))$.

First, we simplify the expression $\pi_2(\mathbf{x}) - \pi_1(\mathbf{x})$.

$$\begin{aligned}
\pi_2(\mathbf{x}) - \pi_1(\mathbf{x}) &= (\pi_2(\mathbf{x}) - \mathbf{x}) - (\pi_1(\mathbf{x}) - \mathbf{x}) \\
&= \left(\frac{|\mathbf{c}|^2 - \mathbf{x} \cdot \mathbf{c}_2}{1 - \mathbf{x} \cdot \mathbf{c}_2}\right)\left(\frac{\mathbf{c}_2}{|\mathbf{c}|^2} - \mathbf{x}\right) - \left(\frac{|\mathbf{c}|^2 - \mathbf{x} \cdot \mathbf{c}_1}{1 - \mathbf{x} \cdot \mathbf{c}_1}\right)\left(\frac{\mathbf{c}_1}{|\mathbf{c}|^2} - \mathbf{x}\right) \\
&= \left(\frac{|\mathbf{c}|^2 - \mathbf{x} \cdot \mathbf{c}_2}{1 - \mathbf{x} \cdot \mathbf{c}_2}\right)\left(\left(\frac{\mathbf{c}_2}{|\mathbf{c}|^2} - \mathbf{x}\right) - \left(\frac{\mathbf{c}_1}{|\mathbf{c}|^2} - \mathbf{x}\right)\right) \\
&= \frac{|\mathbf{c}|^2 - \mathbf{x} \cdot \mathbf{c}_2}{|\mathbf{c}|^2(1 - \mathbf{x} \cdot \mathbf{c}_2)}(\mathbf{c}_2 - \mathbf{c}_1)
\end{aligned}$$

The benefit of this is that it will allow us to remove all other $\mathbf{c}_2 - \mathbf{c}_1$ terms from the exterior product, in effect allowing us to replace all $\mathbf{c}_2$ terms with $\mathbf{c}_1$ terms.

Applying this, we compute $(\mathbf{c}_2 - \mathbf{c}_1) \wedge (\pi_1(\mathbf{x}) - \mathbf{x})$.

$$\begin{aligned}
(\mathbf{c}_2 - \mathbf{c}_1) \wedge (\pi_1(\mathbf{x}) - \mathbf{x}) &= (\mathbf{c}_2 - \mathbf{c}_1) \wedge \left(\left(\frac{|\mathbf{c}|^2 - \mathbf{x} \cdot \mathbf{c}_1}{1 - \mathbf{x} \cdot \mathbf{c}_1}\right)\left(\frac{\mathbf{c}_1}{|\mathbf{c}|^2} - \mathbf{x}\right)\right) \\
&= (\mathbf{c}_2 - \mathbf{c}_1) \wedge \left(\left(\frac{|\mathbf{c}|^2 - \mathbf{x} \cdot \mathbf{c}_1}{1 - \mathbf{x} \cdot \mathbf{c}_1}\right)\left(\frac{\mathbf{c}_1}{|\mathbf{c}|^2} - \frac{\mathbf{c}_1 + \mathbf{c}_2}{2|\mathbf{c}|^2}\right)\right) \\
&= (\mathbf{c}_2 - \mathbf{c}_1) \wedge \left(\left(\frac{|\mathbf{c}|^2 - \mathbf{x} \cdot \mathbf{c}_1}{1 - \mathbf{x} \cdot \mathbf{c}_1}\right)\left(\frac{\mathbf{c}_1 - \mathbf{c}_2}{2|\mathbf{c}|^2}\right)\right) \\
&= \mathbf{0}
\end{aligned}$$

Finally, we compute $\bigwedge_{i=2}^{n+1} (\pi_i(\mathbf{x}) - \pi_1(\mathbf{x}))$ and show that it is a positive scalar multiple of $\bigwedge_{i=2}^{n+1} (\mathbf{c}_i - \mathbf{c}_1)$.

$$
\begin{aligned}
\bigwedge_{i=2}^{n+1}(\pi_i(\mathbf{x})-\pi_1(\mathbf{x})) &= (\pi_2(\mathbf{x})-\pi_1(\mathbf{x}))\wedge\bigwedge_{i=3}^{n+1}(\pi_i(\mathbf{x})-\pi_1(\mathbf{x}))\\
&= \frac{|\mathbf{c}|^2-\mathbf{x}\cdot\mathbf{c}_2}{|\mathbf{c}|^2(1-\mathbf{x}\cdot\mathbf{c}_2)}(\mathbf{c}_2-\mathbf{c}_1)\wedge\bigwedge_{i=3}^{n+1}((\pi_i(\mathbf{x})-\mathbf{x})-(\pi_1(\mathbf{x})-\mathbf{x}))\\
&= \frac{|\mathbf{c}|^2-\mathbf{x}\cdot\mathbf{c}_2}{|\mathbf{c}|^2(1-\mathbf{x}\cdot\mathbf{c}_2)}(\mathbf{c}_2-\mathbf{c}_1)\wedge\bigwedge_{i=3}^{n+1}(\pi_i(\mathbf{x})-\mathbf{x})\\
&= \frac{|\mathbf{c}|^2-\mathbf{x}\cdot\mathbf{c}_2}{|\mathbf{c}|^2(1-\mathbf{x}\cdot\mathbf{c}_2)}(\mathbf{c}_2-\mathbf{c}_1)\wedge\bigwedge_{i=3}^{n+1}\left(\left(\frac{|\mathbf{c}|^2-\mathbf{x}\cdot\mathbf{c}_i}{1-\mathbf{x}\cdot\mathbf{c}_i}\right)\left(\frac{\mathbf{c}_i}{|\mathbf{c}|^2}-\mathbf{x}\right)\right)\\
&= \left(\prod_{i=2}^{n+1}\frac{|\mathbf{c}|^2-\mathbf{x}\cdot\mathbf{c}_i}{|\mathbf{c}|^2(1-\mathbf{x}\cdot\mathbf{c}_i)}\right)(\mathbf{c}_2-\mathbf{c}_1)\wedge\bigwedge_{i=3}^{n+1}\left(\mathbf{c}_i-|\mathbf{c}|^2\mathbf{x}\right)\\
&= \left(\prod_{i=2}^{n+1}\frac{|\mathbf{c}|^2-\mathbf{x}\cdot\mathbf{c}_i}{|\mathbf{c}|^2(1-\mathbf{x}\cdot\mathbf{c}_i)}\right)(\mathbf{c}_2-\mathbf{c}_1)\wedge\bigwedge_{i=3}^{n+1}\left(\mathbf{c}_i-\frac{\mathbf{c}_1+\mathbf{c}_2}{2}\right)\\
&= \left(\prod_{i=2}^{n+1}\frac{|\mathbf{c}|^2-\mathbf{x}\cdot\mathbf{c}_i}{|\mathbf{c}|^2(1-\mathbf{x}\cdot\mathbf{c}_i)}\right)(\mathbf{c}_2-\mathbf{c}_1)\wedge\bigwedge_{i=3}^{n+1}\left(\mathbf{c}_i-\frac{\mathbf{c}_1+\mathbf{c}_1}{2}\right)\\
&= \left(\prod_{i=2}^{n+1}\frac{|\mathbf{c}|^2-\mathbf{x}\cdot\mathbf{c}_i}{|\mathbf{c}|^2(1-\mathbf{x}\cdot\mathbf{c}_i)}\right)\bigwedge_{i=2}^{n+1}(\mathbf{c}_i-\mathbf{c}_1)
\end{aligned}
$$

Since $\mathbf{x}\cdot\mathbf{c}_i<|\mathbf{c}|^2$ for all i, we have that $\prod_{i=2}^{n+1}\frac{|\mathbf{c}|^2-\mathbf{x}\cdot\mathbf{c}_i}{|\mathbf{c}|^2(1-\mathbf{x}\cdot\mathbf{c}_i)}>0$ so the simplices $\pi_1(\mathbf{x})\cdots\pi_{n+1}(\mathbf{x})$ and $\mathbf{c}_1\cdots\mathbf{c}_{n+1}$ have the same orientation. □

Theorem 3.8. *Let $\mathcal{P}$ be a symmetric cocompact arrangement of hyperplanes in $\mathbb{H}^n$ Then almost every point of $\mathbb{H}^n$ is in exactly one open n-dimensional Delaunay cell. In other words, the Delaunay cells tile $\mathbb{H}^n$.*

Proof. From [Prz10], we already know that almost every point is in a total of one open Delaunay cell, although we're counting negatively oriented cells negatively, and we're allowing for a cell to be counted multiple times if it's present multiple times. However, by Proposition 3.7, there are no negatively oriented cells. In order for the total to be one, there must also be no repeated cells of dimension n. □

4. Density in an orthoscheme

We start with some general results about simplices and then move to some more specific results about orthoschemes. For the moment, we make no mention of packings, but will nonetheless define a notion of density. In the following section, we will use this notion of density to study packings of collars about hyperplanes in hyperbolic space. We will often need to assign a notion of density to objects which are of lower dimension than the space in which they lie.

Definition 4.1. Let f be a function $f : \mathbb{H}^n \to (0, 1]$. Let $A_0A_1 \cdots A_k \subset \mathbb{H}^n$ be a k-dimensional simplex. Let H be a $(k-1)$-dimensional flat in the affine hull of $A_0 \cdots A_k$. At each point in $\mathbb{H}^n$, let R denote the distance to H. We define the density of f over $A_0 \cdots A_k$ relative to H to be

$$\delta(A_0 \cdots A_k, H, f) = \frac{\int\limits_{A_0 \cdots A_k} \sinh^{n-k} R \, dV_k}{\int\limits_{A_0 \cdots A_k} f \sinh^{n-k} R \, dV_k}$$

Usually (though not always), we will choose f and H so f is rotationally symmetric about H and so H does not intersect the relative interior of $A_0 \cdots A_k$. With these restrictions, δ could be computed by rotating $A_0 \cdots A_k$ about H to form an n-dimensional object S and then computing $\delta = \frac{\int_S dV_n}{\int_S f \, dV_n}$. This is because rotating any point in $A_0 \cdots A_k$ about H will produce an $(n-k)$-dimensional sphere of radius R. The $(n-k)$-dimensional volume of a sphere is proportional to $\sinh^{n-k} R$. The quotient will cancel the proportionality constant.

Definition 4.2. Let f, $A_0 \cdots A_k \subset \mathbb{H}^n$, H, and R be defined as in Definition 4.1. Let P be a point on edge $A_{k-1}A_k$. Define $v(P) = \int\limits_{A_0 \cdots A_{k-1}P} f \sinh^{n-k} R \, dV_k$. The function $v : A_{k-1}A_k \to [0, v(A_k)]$ increases as P moves from A_{k-1} to A_k so is a bijection. Through an abuse of notation, we thus also define its inverse function to be $P : [0, v(A_k)] \to A_{k-1}A_k$.

Proposition 4.3. *Let f, $A_0 \cdots A_k \subset \mathbb{H}^n$, and H be defined as in Definition 4.1, with $k \geq 2$. In addition, assume that H contains the simplex $A_0 \cdots A_{k-2}$ and is disjoint from the relative interior of $A_0 \cdots A_k$. Let H' be the affine hull of $A_0 \cdots A_{k-2}$. Then*

$$\delta(A_0 \cdots A_k, H, f) = \frac{1}{v(A_k)} \int\limits_0^{v(A_k)} \delta(A_0 \cdots A_{k-2}P(v), H', f) \, dv$$

Proof. As usual, let R denote the distance from a point to H. Within the affine hull of $A_0 \cdots A_k$, we may establish coordinates akin to cylindrical coordinates, with ρ representing the distance to H' and θ representing the amount of rotation about H' from (one half of) H. At neither A_{k-1} nor A_k can ρ equal zero. Also the θ values at A_{k-1} and A_k can't be the same (or else $A_0 \cdots A_k$ wouldn't be k-dimensional). Without loss of generality, we may assume that θ is positive and increasing along edge $A_{k-1}A_k$ as we move from A_{k-1} to A_k.

For any point in $A_0 \cdots A_k$, if we drop altitudes to H and H' and then connect their basepoints, we will form a right triangle (Figure 1). The hypotenuse has length ρ and one of the other edges has length R. The angle opposite the edge of length R has size θ (or $\pi - \theta$ if $\theta > \frac{\pi}{2}$).

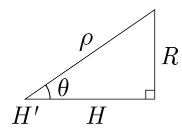

FIGURE 1

From the hyperbolic law of sines, we have that $\sinh R = \sin\theta \sinh\rho$. Let Θ be the value of θ at $P(v)$.

With v and P as in Definition 4.2, we have that $v = \int\limits_{A_0 \cdots A_{k-1} P(v)} f \sinh^{n-k} R \, dV_k$.

Naturally, $\frac{dv}{dv} = 1$. By decomposing $A_0 \cdots A_{k-1} P(v)$ into cross sections of the form θ =constant, we can also determine that

$$1 = \frac{dv}{dv} = \int\limits_{A_0 \cdots A_{k-2} P(v)} \left(f \sinh^{n-k} R \right) \left(\frac{d\Theta}{dv} \sinh\rho \right) dV_{k-1}$$

Within $A_0 \cdots A_{k-2} P(v)$, θ is fixed. We may factor $\frac{d\Theta}{dv}$ out of the integral, producing

$$\frac{d\Theta}{dv} = \frac{1}{\int\limits_{A_0 \cdots A_{k-2} P(v)} f \sinh^{n-k} R \sinh\rho \, dV_{k-1}}$$

Similarly, we can compute that the derivative of $v\delta(A_0 \cdots A_{k-1}P(v), H, f)$ is

$$\begin{aligned}
\frac{d(v\delta(A_0 \cdots A_{k-1}P(v), H, f))}{dv} &= \frac{d}{dv}\left(\int\limits_{A_0 \cdots A_{k-1}P(v)} \sinh^{n-k} R \, dV_k\right) \\
&= \int\limits_{A_0 \cdots A_{k-2}P(v)} \left(\sinh^{n-k} R\right)\left(\frac{d\Theta}{dv}\sinh\rho\right) dV_{k-1} \\
&= \frac{d\Theta}{dv} \int\limits_{A_0 \cdots A_{k-2}P(v)} \sinh^{n-k} R \sinh\rho \, dV_{k-1} \\
&= \frac{\int\limits_{A_0 \cdots A_{k-2}P(v)} \sinh^{n-k} R \sinh\rho \, dV_{k-1}}{\int\limits_{A_0 \cdots A_{k-2}P(v)} f \sinh^{n-k} R \sinh\rho \, dV_{k-1}} \\
&= \frac{\sin^{n-k}\Theta \int\limits_{A_0 \cdots A_{k-2}P(v)} \sinh^{n-k} \rho \sinh\rho \, dV_{k-1}}{\sin^{n-k}\Theta \int\limits_{A_0 \cdots A_{k-2}P(v)} f \sinh^{n-k} \rho \sinh\rho \, dV_{k-1}} \\
&= \frac{\int\limits_{A_0 \cdots A_{k-2}P(v)} \sinh^{n-k+1} \rho \, dV_{k-1}}{\int\limits_{A_0 \cdots A_{k-2}P(v)} f \sinh^{n-k+1} \rho \, dV_{k-1}} \\
&= \delta(A_0 \cdots A_{k-2}P(v), H', f)
\end{aligned}$$

Since $\lim\limits_{v \to 0^+} v\delta(A_0 \cdots A_{k-1}P(v), H, f) = 0$, we have that

$$v(A_k)\delta(A_0 \cdots A_{k-1}A_k, H, f) = \int\limits_0^{v(A_k)} \delta(A_0 \cdots A_{k-2}P(v), H', f) \, dv$$

so

$$\delta(A_0 \cdots A_{k-1}A_k, H, f) = \frac{1}{v(A_k)} \int\limits_0^{v(A_k)} \delta(A_0 \cdots A_{k-2}P(v), H', f) \, dv$$

$\square$

Corollary 4.4. *For Q on edge $A_{k-1}A_k$,*

$$\delta(A_0 \cdots A_{k-1}Q, H, f) = \frac{1}{v(Q)} \int\limits_0^{v(Q)} \delta(A_0 \cdots A_{k-2}P(v), H', f) \, dv$$

and

$$\delta(A_0 \cdots A_{k-2}QA_k, H, f) = \frac{1}{v(A_k) - v(Q)} \int\limits_{v(Q)}^{v(A_k)} \delta(A_0 \cdots A_{k-2}P(v), H', f)\, dv$$

The order of the vertices in a simplex is irrelevant, so the fact that in the preceding results we chose a special role for edge $A_{k-1}A_k$ rather than some other edge is of no significance.

Definition 4.5. For the remainder of this section, we will assume that there is some point $O \in \mathbb{H}^n$ such that $f : \mathbb{H}^n \to (0, 1]$ is an increasing function of the distance to O, and thus is rotationally symmetric about O.

Proposition 4.6. *Let $A_1 \neq O$ be a point in $\mathbb{H}^n$. Let P be a point on edge OA_1. Then the following functions are all decreasing as a function of $\overline{OP}$.*

(1) $\delta(OP, \{O\}, f)$
(2) $\delta(PA_1, \{O\}, f)$
(3) $\delta(OP, \{A_1\}, f)$
(4) $\delta(PA_1, \{A_1\}, f)$

Proof. For $0 \leq t \leq \overline{OA_1}$, let $Q(t)$ be the point on OA_1 which is at a distance of t to O. Then

$$\begin{aligned}\delta(OP, \{O\}, f) &= \frac{\int\limits_{OP} \sinh^{n-1} R\, dV_1}{\int\limits_{OP} f \sinh^{n-1} R\, dV_1} \\ &= \frac{\int\limits_0^{\overline{OP}} \sinh^{n-1} t\, dt}{\int\limits_0^{\overline{OP}} f(Q(t)) \sinh^{n-1} t\, dt}\end{aligned}$$

By a Mean Value Theorem argument, this is a decreasing function of $\overline{OP}$.

The proofs for the other three functions are similar. □

Orthoschemes are common objects for studying problems related to volume. They were used by Rogers, Böröczky, and Florian for proving upper bounds on density for ball packings [BF64, Bör78, Rog58]. Likewise, we will use orthoschemes to place an upper bound on packings of radius r collars about hyperplanes in hyperbolic space.

Definition 4.7. An orthoscheme in $\mathbb{H}^n$ is a simplex $A_0A_1 \cdots A_k$, for $k \leq n$ with the property that for all i, the affine hull of $A_0 \cdots A_i$ is perpendicular to the affine hull of $A_i \cdots A_k$.

Remark 1. Note that for a simplex, the order in which we list the vertices is irrelevant, but for an orthoscheme, the vertex order matters.

Proposition 4.8. *Let $A_0 \cdots A_k$ be an orthoscheme in $\mathbb{H}^n$ with the restriction that within its affine hull, the point which is closest to O is A_0. Choose $i < k$ and let P be a point on A_iA_{i+1}. Let H be either the affine hull of $A_0 \cdots A_{i-1}A_{i+1} \cdots A_k$ or the affine hull of $A_0 \cdots A_iA_{i+2} \cdots A_k$. Then as P moves from A_i to A_{i+1}, the functions $\delta(A_0 \cdots A_iPA_{i+2} \cdots A_k, H, f)$ and $\delta(A_0 \cdots A_{i-1}PA_{i+1} \cdots A_k, H, f)$ are decreasing:*

Proof. Proposition 4.6 proves the result in the case that $k = 1$ and $A_0 = O$. If $k = 1$ and $A_0 \neq O$, then define $g : \mathbb{H}^n \to (0, 1]$ so g and f agree on the affine hull of A_0A_1 and so g is rotationally symmetric about A_0. Since g and f agree on A_0A_1, we may compute δ using g instead of f. Again, Proposition 4.6 proves the result.

We proceed by induction on k.

Let R be the distance to H. As in Definition 4.2, let $v(P) = \int\limits_{A_0 \cdots A_iPA_{i+2} \cdots A_k} f \sinh^{n-k} R \, dV_k$.
Let H' be the affine hull of $A_0 \cdots A_{i-1}A_{i+2} \cdots A_k$. Then from Corollary 4.4, we have that

$$\delta(A_0 \cdots A_iPA_{i+2} \cdots A_k, H, f) = \frac{1}{v(P)} \int\limits_0^{v(P)} \delta(A_0 \cdots A_{i-1}P(v)A_{i+2} \cdots A_k, H', f) \, dv$$

and

$$\delta(A_0 \cdots A_{i-1}PA_{i+1} \cdots A_k, H, f) = \frac{1}{v(A_{i+1}) - v(P)} \int\limits_{v(P)}^{v(A_{i+1})} \delta(A_0 \cdots A_{i-1}P(v)A_{i+2} \cdots A_k, H', f) \, dv$$

Then it is enough to show that $\delta(A_0 \cdots A_{i-1}PA_{i+2} \cdots A_k, H', f)$ decreases as P moves from A_i to A_{i+1}. We break that proof into three cases, depending on the value of i.

Case 1: Assume that $i = 0$. Then $\delta(A_0 \cdots A_{i-1}PA_{i+2} \cdots A_k, H', f) = \delta(PA_2 \cdots A_k, H', f)$ where H' is the affine hull of $A_2 \cdots A_k$. Let P_1 and P_2 be two points on A_0A_1, with P_1 closer to A_0 than P_2 is to A_0 (Figure 2). Let P_2' be P_2 rotated about H' so it lies within $P_1A_2 \cdots A_k$ (Figure 3). As we rotate $P_2A_2 \cdots A_k$ about H' toward $P_1A_2 \cdots A_k$, each point will get

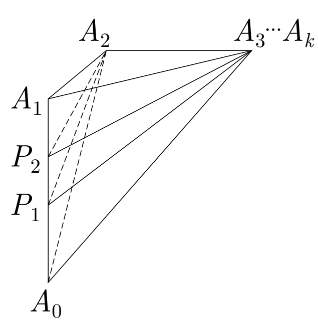

FIGURE 2

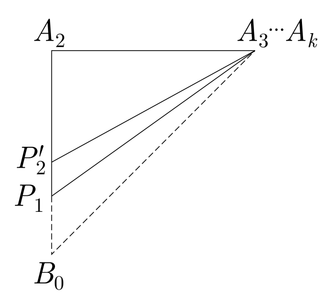

FIGURE 3

closer to A_0, and thus closer to O, so the values of f will decrease. Thus, $\delta(P_2'A_2\cdots A_k, H', f) > \delta(P_2A_2\cdots A_k, H', f)$. If we extend edge A_2P_2' past P_2' and even past P_1, we will eventually find the point B_0 on the affine hull of $P_1A_2\cdots A_k$ which is closest to O. Further, $B_0A_2\cdots A_k$ is an orthoscheme and the points P_1 and P_2' lie on edge B_0A_2 with P_1 closer to B_0 than P_2' is. Thus, by the induction hypothesis, $\delta(P_1A_2\cdots A_k, H', f) > \delta(P_2'A_2\cdots A_k, H', f) > \delta(P_2A_2\cdots A_k, H', f)$.

Case 2: Assume that $i = k-1$. Then $\delta(A_0\cdots A_{i-1}PA_{i+2}\cdots A_k, H', f) = \delta(A_0\cdots A_{k-2}P, H', f)$, where H' is the affine hull of $A_0\cdots A_{k-2}$. Let P_1 and P_2 be two points on $A_{k-1}A_k$, with P_1 closer to A_{k-1} than P_2 is (Figure 4). Let P_1' be P_1 rotated about H' into $A_0\cdots A_{k-2}P_2$ (Figure 5).

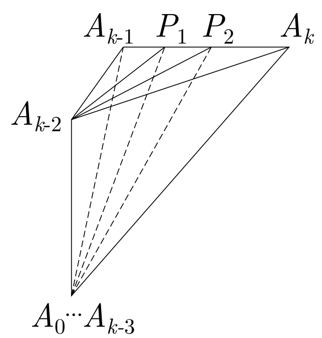

FIGURE 4

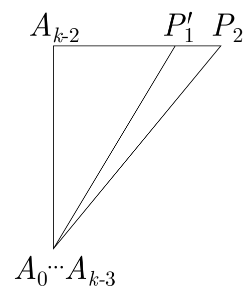

FIGURE 5

Within the affine hull of $A_0\cdots A_k$, rotation about H' won't change the value of f, so $\delta(A_0\cdots A_{k-2}P_1, H', f) = \delta(A_0\cdots A_{k-2}P_1', H', f)$. By the induction hypothesis applied to the orthoscheme $A_0\cdots A_{k-2}P_2$, we have that $\delta(A_0\cdots A_{k-2}P_2, H', f) < \delta(A_0\cdots A_{k-2}P_1', H', f)$.

Case 3: Assume that $0 < i < k-1$. Let P_1 and P_2 be points on A_iA_{i+1}, with P_1 closer to A_i than P_2 is (Figure 6). Let P_2' be P_2 rotated about H' into the affine hull of $A_0\cdots A_{i-1}P_1A_{i+2}\cdots A_k$ (Figure 7 - showing one more dimension than Figure 6). The points P_1 and P_2 both project into the affine

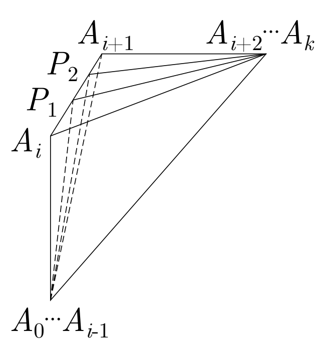

FIGURE 6

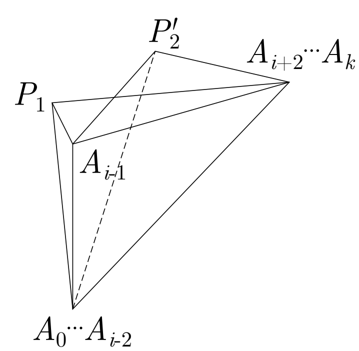

FIGURE 7

hull of $A_0 \cdots A_{i-1}$ at A_{i-1} and project into the affine hull of $A_{i+2} \cdots A_k$ at A_{i+2}. The same will be true of the point P_2'. Within the affine hull of $A_0 \cdots A_{i-1}P_1A_{i+2} \cdots A_k$, any point having that property must lie within the affine hull of $A_{i-1}P_1A_{i+2}$. Thus, A_{i-1}, A_{i+2}, P_1, and P_2' are coplanar.

The affine hull of P_1A_{i+2} is closer to A_{i-1} than the affine hull of $P_2'A_{i+2}$ is. Also, the affine hull of $A_{i-1}P_2'$ is closer to A_{i+2} than the affine hull of $A_{i-1}P_1$ is. Thus, the edges $A_{i-1}P_2'$ and P_1A_{i+2} intersect at some point Q (Figure 8). Note that $A_0 \cdots A_{i-1}P_1A_{i+2} \cdots A_k$ and $A_0 \cdots A_{i-1}P_2'A_{i+2} \cdots A_k$

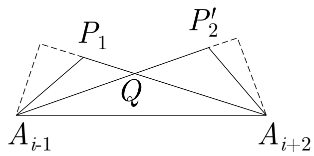

FIGURE 8

aren't necessarily orthoschemes. Applying the induction hypothesis again,

$$\begin{aligned}\delta(A_0 \cdots A_{i-1}P_1A_{i+2} \cdots A_k, H', f) &> \delta(A_0 \cdots A_{i-1}QA_{i+2} \cdots A_k, H', f)\\ &> \delta(A_0 \cdots A_{i-1}P_2'A_{i+2} \cdots A_k, H', f)\\ &= \delta(A_0 \cdots A_{i-1}P_2A_{i+2} \cdots A_k, H', f)\end{aligned}$$

□

Theorem 4.9. *Let $A_0 \cdots A_n$ and $B_0 \cdots B_n$ be two orthoschemes in $\mathbb{H}^n$, with $A_0 = B_0 = O$. Let H be any hyperplane in $\mathbb{H}^n$. If $\overline{A_iA_{i+1}} \leq \overline{B_iB_{i+1}}$ for all $i < n$, then $\delta(A_0 \cdots A_n, H, f) \geq \delta(B_0 \cdots B_n, H, f)$.*

Proof. At any given point, let R be the distance to H. Since $\sinh^{n-n} R = \sinh^0 R = 1$, the value of R is irrelevant, and thus the choice of H is also irrelevant.

Let m be the smallest i for which $\overline{A_iA_{i+1}} < \overline{B_iB_{i+1}}$. If $m = n-1$, then a direct application of Proposition 4.8 completes the proof.

Assume that $m < n-1$. Rotate $B_0 \cdots B_n$ about O so $A_i = B_i$ for all $i \leq m$. Rotate $B_0 \cdots B_n$ about the affine hull of $A_0 \cdots A_m$ so $A_0 \cdots A_{m+1}B_{m+2} \cdots B_n$ is an orthoscheme. As in the preceding proof, this forces A_m, A_{m+1}, B_{m+1} and B_{m+2} to be coplanar. $A_mA_{m+1}B_{m+2}$ and $A_mB_{m+1}B_{m+2}$ are both orthoschemes (right triangles). If they lie on opposite sides of A_mB_{m+2}, we may reflect so they lie on the same side. Then $A_{m+1}B_{m+2}$ and A_mB_{m+1} intersect. Call their intersection point Q. Let H be the affine hull of $A_0 \cdots A_mB_{m+2} \cdots B_n$. Applying Proposition 4.8,

$$\begin{aligned}\delta(A_0 \cdots A_mA_{m+1}B_{m+2} \cdots B_n, H, f) &> \delta(A_0 \cdots A_mQB_{m+2} \cdots B_n, H, f)\\ &> \delta(A_0 \cdots A_mB_{m+1}B_{m+2} \cdots B_n, H, f)\\ &= \delta(B_0 \cdots B_mB_{m+1}B_{m+2} \cdots B_n, H, f)\\ &= \delta(B_0 \cdots B_n, H, f)\end{aligned}$$

By following this process, we have made one more of the B_i equal to the corresponding A_i. Edge $B_m B_{m+1}$ has been replaced by edge $A_m A_{m+1}$. Edge $B_{m+1}B_{m+2}$ has been replaced by edge $A_{m+1}B_{m+2}$, which is even longer, so is still at least as long as edge $A_{m+1}A_{m+2}$. None of the other $\overline{B_i B_{i+1}}$ has been changed. Thus, we may repeat the process, comparing orthoscheme $A_0 \cdots A_n$ to orthoscheme $A_0 \cdots A_m A_{m+1} B_{m+2} \cdots B_n$. At each step, the density increases. We might need to change the choice of H at each step, but the choice of H is irrelevant, so we are free to do this. □

Proposition 4.10. *Let $A_0 \cdots A_k B$ be an orthoscheme in $\mathbb{H}^n$, with $A_0 = O$. Let H be the affine hull of $A_0 \cdots A_k$ and let H' be the affine hull of $A_0 \cdots A_{k-1}$. Then $\delta(A_0 \cdots A_k B, H, f) > \delta(A_0 \cdots A_{k-1} B, H', f)$.*

Proof. Let P be a point on $A_k B$ and let v be as in Definition 4.2. Then by Corollary 4.4,

$$\delta(A_0 \cdots A_k B, H, f) = \frac{1}{v(B)} \int_0^{v(B)} \delta(A_0 \cdots A_{k-1} P(v), H', f)\, dv$$

We have that $A_0 \cdots A_{k-1} P(v)$ is an orthoscheme and that $\overline{A_{k-1} P(v)}$ increases with v. The density $\delta(A_0 \cdots A_{k-1} P(v), H', f)$ is a decreasing function of v, by Proposition 4.8. The Mean Value Theorem finds some $\tilde{v} \in (0, v(B))$ for which

$$\delta(A_0 \cdots A_k B, H, f) = \delta(A_0 \cdots A_{k-1} P(\tilde{v}), H', f) > \delta(A_0 \cdots A_{k-1} B, H', f)$$

□

Corollary 4.11. *Let $A_0 \cdots A_k B$ be an orthoscheme in $\mathbb{H}^n$, with $A_0 = O$. Choose some $i < k$. Let H be the affine hull of $A_0 \cdots A_k$ and let H' be the affine hull of $A_0 \cdots A_i$. Then $\delta(A_0 \cdots A_k B, H, f) > \delta(A_0 \cdots A_i B, H', f)$.*

Proof. Start with $i = k - 1$. Use the preceding proposition to repeatedly decrease i by 1, removing a vertex from the orthoscheme each time. □

5. Density of a packing of collars

In this section, we prove an upper bound on density for packings of radius r collars about hyperplanes in $\mathbb{H}^n$. We first need to determine some of the consequences of hyperplanes being a distance of $2r$ from each other.

Proposition 5.1. *Let $\Pi_1, \cdots, \Pi_m$ be disjoint hyperplanes in $\mathbb{H}^n$ each of which is at a distance of at least $2r$ to all of the others. If $v \in \mathbb{H}^n$ is a point which is equidistant from all of the Π_i, then the distance from v to any of the Π_i is at least $\cosh^{-1}\left(\sqrt{\frac{2(m-1)}{m}} \cosh r\right)$. This value is achievable only if each of $\Pi_1, \cdots, \Pi_m$ is at a distance of exactly $2r$ to all of the others and v lies in their closed Delaunay cell.*

Proof. Represent each Π_i as a vector $\mathbf{c}_i \in D^n$, as in Definition 2.1. Without loss of generality, we may place v at $\mathbf{0} \in D^n$. Then $|\mathbf{c}_i|$ is independent of i. To simplify the notation, we denote $|\mathbf{c}_i|$ as $|\mathbf{c}|$. From Lemma 3.1, we have that for $i \neq j$, $\cosh 2r \leq \frac{|\mathbf{c}|^4 - \mathbf{c}_i \cdot \mathbf{c}_j}{|\mathbf{c}|^2(1-|\mathbf{c}|^2)}$, so $\mathbf{c}_i \cdot \mathbf{c}_j \leq |\mathbf{c}|^4 - |\mathbf{c}|^2(1-|\mathbf{c}|^2)\cosh 2r$.

$$\begin{aligned} 0 &\leq \left|\sum_{i=1}^{m} \mathbf{c}_i\right|^2 \\ &= \sum_{i=1}^{m} |\mathbf{c}_i|^2 + 2\sum_{i<j} \mathbf{c}_i \cdot \mathbf{c}_j \\ &\leq m|\mathbf{c}|^2 + m(m-1)(|\mathbf{c}|^4 - |\mathbf{c}|^2(1-|\mathbf{c}|^2)\cosh 2r) \end{aligned}$$

Rearranging the inequality, we get that $|\mathbf{c}|^2 \geq \frac{(m-1)\cosh 2r - 1}{(m-1)(1+\cosh 2r)}$. Then the distance from v to any of the Π_i is

$$\tanh^{-1}|\mathbf{c}| \geq \tanh^{-1}\sqrt{\frac{(m-1)\cosh 2r - 1}{(m-1)(1+\cosh 2r)}} = \cosh^{-1}\left(\sqrt{\frac{2(m-1)}{m}}\cosh r\right)$$

Since equality comes only when all of the hyperplanes are at a distance of exactly $2r$ to each other and $\sum_{i=1}^{m} \mathbf{c}_i = \mathbf{0}$, we see that the $\mathbf{c}_i$ form a regular simplex, with $\mathbf{0}$ at its center. $\square$

We define $l_m(r)$ to be $l_m(r) = \cosh^{-1}\left(\sqrt{\frac{2(m-1)}{m}}\cosh r\right)$.

Proposition 5.2. *For $m \leq n$, let $\Pi_1, \cdots, \Pi_m, \Pi_{m+1}$ be disjoint hyperplanes in $\mathbb{H}^n$ each of which is at a distance of at least $2r$ to all of the others. Let F be the set of points which are equidistant from all of $\Pi_1, \cdots, \Pi_m$. We assume that F is nonempty. Let B be the point on F which is closest to Π_1 (and hence to $\Pi_2, \cdots, \Pi_m$). If B is at least as close to Π_{m+1} as it is to Π_1, then the distance from B to any of $\Pi_1, \cdots, \Pi_m$ is at least $\cosh^{-1}\left(\sqrt{2}\cosh r\right)$.*

Proof. Place B at the origin in D^n. Represent each of the hyperplanes Π_i by a vector $\mathbf{c}_i$, as in Definition 2.1. Then $|\mathbf{c}_1| = \cdots = |\mathbf{c}_m| \geq |\mathbf{c}_{m+1}|$. Also, $\mathbf{0}$ lies in the affine hull of $\mathbf{c}_1, \cdots, \mathbf{c}_m$, so $\mathbf{c}_1, \cdots, \mathbf{c}_m$ are linearly dependent. The set F is the orthogonal subspace to the span of $\mathbf{c}_1, \cdots, \mathbf{c}_m$.

Without loss of generality, $\mathbf{c}_1, \cdots, \mathbf{c}_m \in \mathbb{R}^{m-1} \subset \mathbb{R}^n$ and $\mathbf{c}_{m+1} \in \mathbb{R}^m \subseteq \mathbb{R}^n$. Further, we may assume that the m^{th} coordinate of $\mathbf{c}_{m+1}$ is nonnegative.

If we increase the m^{th} coordinate of $\mathbf{c}_{m+1}$, that will increase $|\mathbf{c}_{m+1}|$, but won't change the value of $\mathbf{c}_i \cdot \mathbf{c}_{m+1}$, for $i \leq m$. It's not difficult to prove that if $\mathbf{c}_i \cdot \mathbf{c}_{m+1}$ is fixed and $|\mathbf{c}_{m+1}|$ is increasing, then $\cosh^{-1} \frac{|\mathbf{c}_i|^2|\mathbf{c}_{m+1}|^2 - \mathbf{c}_i \cdot \mathbf{c}_{m+1}}{|\mathbf{c}_i||\mathbf{c}_{m+1}|\sqrt{1-|\mathbf{c}_i|^2}\sqrt{1-|\mathbf{c}_{m+1}|^2}}$ is increasing, so the distance from Π_i to Π_{m+1} is increasing. Thus, if we increase the m^{th} coordinate of $\mathbf{c}_{m+1}$ until $|\mathbf{c}_{m+1}| = |\mathbf{c}_1|$, we will still satisfy the hypotheses of the result we're trying to prove.

Thus, we may assume that $\mathbf{c}_1, \cdots, \mathbf{c}_{m+1}$ are points on an m-dimensional sphere centered at $\mathbf{0}$ and that $\mathbf{c}_1, \cdots, \mathbf{c}_m$ lie on the equator of that sphere. Place a new point $\mathbf{c}_{m+2}$ at $(0, 0, \cdots, 0, -1)$. We have placed $m+2$ points on an m-dimensional sphere. A result of Rankin [Ran55] proves that some two of these points form a central angle (with vertex at $\mathbf{0}$) which is at most $\frac{\pi}{2}$.

If all such pairs of points include $\mathbf{c}_{m+2}$, then we may perturb all of $\mathbf{c}_1, \cdots, \mathbf{c}_{m+1}$ away from $\mathbf{c}_{m+2}$, resulting in no central angles which are less than or equal to $\frac{\pi}{2}$. This would violate Rankin's result. Thus, we may assume that there are some $i < j < m+2$ such that $\mathbf{c}_i$ and $\mathbf{c}_j$ form a central angle which is less than or equal to $\frac{\pi}{2}$. As a consequence $\mathbf{c}_i \cdot \mathbf{c}_j \geq 0$.

Since $\mathbf{c}_1, \cdots, \mathbf{c}_{m+1}$ all have the same length, we denote that length by $|\mathbf{c}|$. Then $\frac{|\mathbf{c}|^4}{|\mathbf{c}|^2(1-|\mathbf{c}|^2)} \geq \frac{|\mathbf{c}|^4 - \mathbf{c}_i \cdot \mathbf{c}_j}{|\mathbf{c}|^2(1-|\mathbf{c}|^2)} \geq \cosh 2r$. Simplifying this inequality, we have that $\frac{1}{1-|\mathbf{c}|^2} \geq 2\cosh^2 r$, so $\tanh^{-1}|\mathbf{c}| \geq \cosh^{-1}\left(\sqrt{2}\cosh r\right)$. The distance from any of the hyperplanes to $\mathbf{0}$ is $\tanh^{-1}|\mathbf{c}|$, so we have proved the result. $\square$

Considering that $\lim_{m\to\infty} l_m(r) = \cosh^{-1}\left(\sqrt{2}\cosh r\right)$, we call this quantity $l_\infty(r)$. Note that $l_2(r) < l_3(r) < \cdots < l_\infty(r)$.

Definition 5.3. Let Π and V be disjoint hyperplanes in $\mathbb{H}^n$. Let their common perpendicular line meet Π at point A_0 and meet V at point B_0. An orthoprism with base in Π is a polytope $A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}$ satisfying the following conditions.

(1) $B_0 \cdots B_{n-1}$ is an orthoscheme in V.
(2) For all i, the projection of B_i into Π is A_i.

From these conditions, it follows that $A_0 \cdots A_{n-1}$ is an orthoscheme in Π.

Proposition 5.4. *Let $\mathcal{P}$ be an arrangement of hyperplanes in $\mathbb{H}^n$, all of which are separated from each other by a distance of at least $2r$. Assume that each Voronoi cell stays within bounded distance of the hyperplane at its center. Let D be the Voronoi cell associated with hyperplane Π_1. Let F_0 be an $(n-1)$-dimensional face of D and let B_0 be the closest point to Π_1 on the affine hull of F_0. Continuing, let F_i be an $(n-i-1)$-dimensional face of F_{i-1} and let B_i be the closest point to Π_1 on the affine hull of F_i. Let A_i be the projection of B_i into Π_1. Then $A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}$ is an orthoprism with base in Π_1. Further, we have that $\overline{A_i B_i} \geq l_{i+2}(r)$, for all i.*

Proof. Each F_i is a face of the Voronoi cell D, so is equidistant from at least $i+2$ of the hyperplanes in $\mathcal{P}$. Since F_{i+1} is on the boundary of F_i, F_{i+1} is equidistant from at least the same hyperplanes as F_i is. Find hyperplanes $\Pi_1, \cdots, \Pi_{n+1} \in \mathcal{P}$ such that for all i, F_i is equidistant from $\Pi_1, \cdots, \Pi_{i+2}$ but not equidistant from $\Pi_1, \cdots, \Pi_{i+3}$.

For some k, place B_k at the origin in D^n. Represent each of the Π_i as a vector $\mathbf{c}_i \in D^n$. All of the $\mathbf{c}_i$ have the same length. The set of points equidistant from $\Pi_1, \cdots, \Pi_{k+2}$ is the orthogonal subspace to the span of

$\mathbf{c}_1, \cdots, \mathbf{c}_{k+2}$. The point on this flat which is closest to Π_1 is $B_k = \mathbf{0}$. The points $B_{k+1}, \cdots, B_{n-1}$ all lie in this same orthogonal subspace, since they're all equidistant from $\Pi_1, \cdots, \Pi_{k+2}$. For $i < k$, each B_i lies in the span of $\mathbf{c}_1, \cdots, \mathbf{c}_{k+2}$. Then the affine hull of $B_0 \cdots B_k$ is orthogonal to the affine hull of $B_k \cdots B_{n-1}$. This verifies that $B_0 \cdots B_{n-1}$ is an orthoscheme.

By definition, B_0 is the closest point to Π_1 within the $(n-1)$-dimensional affine hull of F_0. Then $A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}$ is an orthoprism.

Each of the B_i is equidistant from $\Pi_1, \cdots, \Pi_{i+2}$, so the distance from B_i to Π_1 is at least $l_{i+2}(r)$. The distance from B_i to Π_1 is exactly $\overline{A_i B_i}$, so $\overline{A_i B_i} \geq l_{i+2}(r)$. □

Definition 5.5. Let $A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}$ be an orthoprism with base in the hyperplane $\Pi \in \mathbb{H}^n$. Choose a positive number $r \leq \overline{A_0 B_0}$. Let C be the collar of radius r about Π. The density of C within the orthoprism is defined to be $\frac{\mathrm{Vol}(C \cap (A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}))}{\mathrm{Vol}(A_0 \cdots A_{n-1} B_0 \cdots B_{n-1})}$.

Proposition 5.6. *Let Π and V be disjoint hyperplanes in $\mathbb{H}^n$. Choose $r > 0$ to be at most the distance from Π to V. Let C be the collar of radius r about Π. Then there is a continuous function $f : \Pi \to (0, 1]$ such that for any orthoprism $A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}$ with base in Π and $B_0, \cdots, B_{n-1} \in V$, we have that the density of C within $A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}$ is given by $\delta(A_0 \cdots A_{n-1}, H, f)$ (computed within the $(n-1)$-dimensional space Π). The $n-2$ dimensional flat H may be chosen arbitrarily within Π. The function f is increasing as a function of distance to A_0 and is invariant under rotation within Π about A_0.*

Proof. There is some function $g : \Pi \to (0, \infty)$ such that the volume of $A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}$ is $\int\limits_{A_0 \cdots A_{n-1}} g \, dV_{n-1}$. The function g increases as a function of distance to A_0 and is invariant under rotation within Π about A_0. There is some increasing function $h : (0, \infty) \to (0, \infty)$ such that the volume of $C \cap (A_0 \cdots A_{n-1} B_0 \cdots B_{n-1})$ is $h(r) \int\limits_{A_0 \cdots A_{n-1}} dV_{n-1}$. Then the density of C within the orthoprism is

$$\frac{h(r) \int\limits_{A_0 \cdots A_{n-1}} dV_{n-1}}{\int\limits_{A_0 \cdots A_{n-1}} g \, dV_{n-1}} = \frac{\int\limits_{A_0 \cdots A_{n-1}} dV_{n-1}}{\int\limits_{A_0 \cdots A_{n-1}} \frac{g}{h(r)} \, dV_{n-1}}$$

Let $f = \frac{g}{h(r)} \leq 1$. The hyperplane Π is isometric to $\mathbb{H}^{n-1}$. Let H be any $(n-2)$-dimensional flat within Π. Let R be the distance from any point in

Π to H. Then within the $(n-1)$-dimensional space Π,

$$\delta(A_0\cdots A_{n-1},H,f) = \frac{\int\limits_{A_0\cdots A_{n-1}} \sinh^{(n-1)-(n-1)} R\,dV_{n-1}}{\int\limits_{A_0\cdots A_{n-1}} \frac{g}{h(r)}\sinh^{(n-1)-(n-1)} R\,dV_{n-1}} = \frac{\int\limits_{A_0\cdots A_{n-1}} dV_{n-1}}{\int\limits_{A_0\cdots A_{n-1}} \frac{g}{h(r)}\,dV_{n-1}}$$

□

Definition 5.7. Let $\Pi_1,\cdots,\Pi_{n+1}$ be hyperplanes in $\mathbb{H}^n$, each of which is at a distance of exactly $2r$ to the others. Let T be their closed Delaunay cell, which is a truncated regular ultra-ideal simplex. Let

$$U = \{x\in T \mid x \text{ is within a distance of } r \text{ to at least one of the } \Pi_i\}$$

Then U is a collar of radius r about the truncation faces of T. Define $d_n(r) = \frac{\mathrm{Vol}(U)}{\mathrm{Vol}(T)}$.

Proposition 5.8. *Let $A_0\cdots A_{n-1}B_0\cdots B_{n-1}$ be an orthoprism in which $\overline{A_mB_m} = l_{m+2}(r)$ for all m. Let C be a collar of radius r about the base of the orthoprism. Then the density of C within the orthoprism is $d_n(r)$.*

Proof. For a fixed r, any two such orthoprisms are congruent. Thus, it is sufficient to prove that one such orthoprism has density $d_n(r)$.

Let $\Pi_1,\cdots,\Pi_{n+1}$ be $n+1$ hyperplanes in $\mathbb{H}^n$, such that distance between any two of them is always $2r$. In D^n, we may represent Π_i as a vector $\mathbf{c}_i$, as in Definition 2.1. Then the vectors $\mathbf{c}_1,\cdots,\mathbf{c}_{n+1}$ form a regular simplex. Further, by Theorem 3.5 the closed Delaunay cell T of $\Pi_1,\cdots,\Pi_{n+1}$ is the ultra-ideal simplex with vertices $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2},\cdots,\frac{\mathbf{c}_{n+1}}{|\mathbf{c}_{n+1}|^2}$ with the (ultra-ideal) vertices truncated by the hyperplanes $\Pi_1,\cdots,\Pi_{n+1}$. The Euclidean simplex with vertices $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2},\cdots,\frac{\mathbf{c}_{n+1}}{|\mathbf{c}_{n+1}|^2}$ may be decomposed into congruent orthoschemes, all of which will have $\mathbf{0}$ as a vertex. If a face of one of these orthoschemes is transverse to Π_i, then that face passes through $\frac{\mathbf{c}_i}{|\mathbf{c}_i|^2}$, so is (hyperbolically) perpendicular to Π_i. Thus, this decomposition of the simplex into congruent orthoschemes induces a decomposition of T into congruent orthoprisms. Thus, the density within the orthoprisms is $d_n(r)$.

All that remain to be proven is that the vertices are at a distance of $l_{m+2}(r)$ from the hyperplanes Π_i. Let $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2}B_0B_1\cdots B_{n-1}$ be one of the orthoschemes (with $B_{n-1}=\mathbf{0}$). Then vertex B_m is (in a Euclidean sense) equidistant from $m+2$ of the $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2},\cdots,\frac{\mathbf{c}_{i+1}}{|\mathbf{c}_{i+1}|^2}$, without loss of generality, $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2},\cdots,\frac{\mathbf{c}_{m+2}}{|\mathbf{c}_{m+2}|^2}$. Further B_m lies in the convex hull of $\frac{\mathbf{c}_1}{|\mathbf{c}_1|^2},\cdots,\frac{\mathbf{c}_{m+2}}{|\mathbf{c}_{m+2}|^2}$. Thus B_m is (hyperbolically) equidistant from $\Pi_1,\cdots,\Pi_{m+2}$ and B_m lies within the affine hull of $\pi_1(B_m),\cdots,\pi_{m+2}(B_m)$.

Then the hyperbolic distance from B_m to Π_1 is $l_{m+2}(r)$. □

Proposition 5.9. *Let $A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}$ be an orthoprism in $\mathbb{H}^n$. Let r be such that $\overline{A_i B_i} \geq l_{i+2}(r)$ for all i. Let C be the collar of radius r about the base of the orthoprism. Then the density of C within $A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}$ is at most $d_n(r)$.*

Proof. Let Π be the affine hull of $A_0 \cdots A_{n-1}$. If $\overline{A_0 B_0} > l_2(r) = r$, then find some hyperplane $V \subset \mathbb{H}^n$ such that

(1) V contains $B_1 \cdots B_{n-1}$
(2) V intersects $A_0 B_0$ at some point $\tilde{B}_0$
(3) the distance from V to Π is r.

Let B_0' be the point in V which is closest to Π and let A_0' be the point in Π which is closest to V (Figure 9). Then $A_0' A_1 \cdots A_{n-1} B_0' B_1 \cdots B_{n-1}$ is an orthoprism. The point A_0 lies on edge $A_0' A_1$ and the point $\tilde{B}_0$ lies on edge $B_0' B_1$.

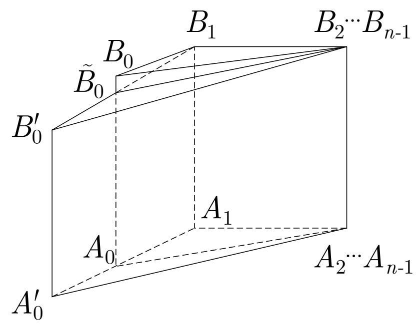

Figure 9

The density of C within $A_0 \cdots A_{n-1} \tilde{B}_0 B_1 \cdots B_{n-1}$ is obviously greater than the density of C within $A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}$. Use Proposition 5.6 to find the function $f : \Pi \to (0, \infty)$ such that the density of C within $A_0' A_1 \cdots A_{n-1} B_0' B_1 \cdots B_{n-1}$ is $\delta(A_0' A_1 \cdots A_{n-1}, H, f)$, where H is an arbitrary $(n-2)$-dimensional flat within Π.

Applying Proposition 4.8, $\delta(A_0' A_1 \cdots A_{n-1}, H, f) > \delta(A_0 A_1 \cdots A_{n-1}, H, f)$. We have found a new orthoprism $A_0' A_1 \cdots A_{n-1} B_0' B_1 \cdots B_{n-1}$ which still satisfies the hypotheses of the proposition we're trying to prove. The collar C has higher density within this new orthoprism than within the original orthoprism $A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}$. Further, $\overline{A_0' B_0'} = l_2(r) = r$.

If $\overline{A_i B_i} = l_{i+2}(r)$ for all $i > 0$, then we have proved the result. Otherwise, find the smallest i for which $\overline{A_i B_i} > l_{i+2}(r)$. If we were to move A_i closer to A_0' (while also moving $B_i \in V$ so as to maintain the requirements for an orthoprism), that would decrease $\overline{A_i B_i}$. Thus, if we applied Theorem 4.9 to decrease $\overline{A_{i-1} A_i}$ (or $\overline{A_0' A_1}$ in the case that $i = 1$) we would decrease $\overline{A_0' A_i}$ and thus also decrease $\overline{A_i B_i}$. While doing this, density increases. We may repeat the application of Theorem 4.9 to ensure that $\overline{A_i B_i} = l_{i+2}(r)$ for all $i > 0$.

At the end of this process, we will have produced an orthoprism of the type described in Proposition 5.8, which will have density at least as large as the original orthoprism. □

Theorem 5.10. *Let $\mathcal{P}$ be an arrangement of hyperplanes in $\mathbb{H}^n$, each of which is at a distance of at least $2r$ to all of the others. Let C be a collar about one of the hyperplanes and let D be the Voronoi cell associated with that hyperplane. Then the density of C within V is at most $d_n(r)$.*

Proof. Let Π_1 be the chosen hyperplane. Let F_0 be an $(n-1)$ dimensional face of D. Then F_0 is equidistant from Π_1 and one other hyperplane in $\mathcal{P}$. Let B_0 be the point closest to Π_1 in the affine hull of F_0. By Proposition 5.2, if $B_0 \notin F_0$, then every point of F_0 is at a distance of at least $l_\infty(r)$ to Π_1. Since $l_\infty(r) > l_{n+1}(r)$, $d_n(r)$ is an upper bound on the density of C within the solid formed by taking the convex hull of F_0 and its projection into Π_1. Thus, we assume that $B_0 \in F_0$. Let A_0 be the projection of B_0 into Π_1.

Assume that we have found $F_0, \cdots, F_k$ and $B_0, \cdots, B_k$ such that

(1) Each F_i is an $(n-i-1)$-dimensional face of D and F_i lies on the boundary of F_{i-1}.
(2) Each B_i is the point closest to Π_1 on the affine hull of F_i.
(3) For all i, $B_i \in F_i$.

Let A_i be the projection of B_i into Π_1. Then for all i, $\overline{A_iB_i} \geq l_{i+2}(r)$.

If $k < n-2$, then choose any $(n-k-2)$-dimensional cell F_{k+1} on the boundary of F_k. Let B_{k+1} be the point closest to Π_1 on the affine hull of F_{k+1}. There are two cases to consider.

Case 1: Assume that every point on F_{k+1} is at a distance of at least $l_\infty(r) > l_{n+1}(r)$ from Π_1. We can decompose the convex hull of $(B_0 \cdots B_k) \cup F_{k+1}$ into two pieces, a piece K consisting of points that are within $l_\infty(r)$ of Π_1 and a piece L consisting of points that are at a distance greater than $l_\infty(r)$ to Π_1. Taking the convex hull of L and its projection into Π_1 produces a region in which C has density less than $d_n(r)$.

Let P be any point on the boundary between K and L. All points on the boundary between K and L are at the same distance to $B_0 \cdots B_k$ as P is. Thus, the region K could be formed by partial rotation of $B_0 \cdots B_k P$ about $B_0 \cdots B_k$.

Let Q be the projection of P into Π_1. If we form a solid body S by taking the convex hull of K and its projection into Π, we could compute the density of C within S by computing $\delta(A_0 \cdots A_k Q, H, f)$ where H is the affine hull of $A_0 \cdots A_k$. For all i, $\overline{A_iB_i} \geq l_{i+2}(r)$. Also, $\overline{PQ} = l_\infty(r) > l_{n+1}(r)$. By Corollary 4.11, the density of C within S is less than the density of f over any orthoscheme $A_0 \cdots A_k A_{k+1} \cdots A_{n-2} Q$. In particular, by choosing A_{k+1} through A_{n-2} to be far enough from A_0, we could form an orthoprism satisfying the hypotheses of Proposition 5.9. Thus, the density of C within S is at most $d_n(r)$.

Case 2: Assume F_{k+1} contains points whose distance to Π_1 is at most $l_\infty(r)$. Then by Proposition 5.2, $B_{k+1} \in F_{k+1}$.

We can continue this process until it terminates. When it terminates, either we've reached the Case 1 (so the density is at most $d_n(r)$) or we've reached $k = n-2$. If $k = n-2$, then find F_{n-1} in the boundary of F_{n-2}. F_{n-1} will be 0-dimensional and might lie on $\partial\mathbb{H}^n$. Of course, $B_{n-1} = F_{n-1}$. Also, if $F_{n-1} \in \mathbb{H}^n$, then F_{n-1} is equidistant from at least $n+1$ of the hyperplanes in $\mathcal{P}$. Let A_{n-1} be the projection of F_{n-1} into Π_1. Regardless of whether B_{n-1} is in $\mathbb{H}^n$ or $\partial\mathbb{H}^n$, $\overline{A_{n-1}B_{n-1}} \geq l_{n+1}(r)$.

We've found $B_0 \cdots B_{n-1}$ satisfying the hypotheses of Proposition 5.4 so $A_0 \cdots A_{n-1} B_0 \cdots B_{n-1}$ is an orthoprism in which $\overline{A_iB_i} \geq l_{i+2}(r)$ for all i. By Proposition 5.9, the density of C within this orthoprism is at most $d_n(r)$.

We can decompose the entire cell D into pieces of the types created in Case 1 or Case 2. In all cases, then density of C within a given piece is at most $d_n(r)$. Thus, the density of C within D is at most $d_n(r)$. $\square$

Corollary 5.11. *For packings of radius r collars about hyperplanes in $\mathbb{H}^n$, the density is at most $d_n(r)$.*

As usual, it possible that for some packings, the density won't be well-defined. Naturally, Corollary 5.11 wouldn't apply in those cases.

For most values of r, there's no reason to believe that this bound is sharp. However, for some values of r, the bound will be sharp. If all of the dihedral angles in the polytope T (constructed in Definition 5.7) are submultiples of π, then we could tile $\mathbb{H}^n$ with copies of T, producing an optimal packing.

Also, although we conjecture that $d_n(r)$ is an increasing function of r, we make no effort to prove it. If $d_n(r)$ is increasing as a function of r, then Böröczky's density bound for horoball packing [Bör78] is also a density bound for packings of collars about hyperplanes.

References

[BF64] K. Böröczky and A. Florian. Über die dichteste Kugelpackung im hyperbolischen Raum. *Acta Math. Acad. Sci. Hungar*, 15:237–245, 1964.

[Bör78] K. Böröczky. Packing of spheres in spaces of constant curvature. *Acta Math. Acad. Sci. Hungar.*, 32:243–326, 1978.

[Fen89] Werner Fenchel. *Elementary geometry in hyperbolic space*, volume 11 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1989. With an editorial by Heinz Bauer.

[MM03] T. H. Marshall and G. J. Martin. Packing strips in the hyperbolic plane. *Proc. Edinb. Math. Soc. (2)*, 46(1):67–73, 2003.

[Prz10] Andrew Przeworski. Delaunay Cells for Arrangements of Flats in Hyperbolic Space, 2010. preprint, available at http://przeworskia.people.cofc.edu.

[Ran55] R. A. Rankin. The closest packing of spherical caps in n dimensions. *Proc. Glasgow Math. Assoc.*, 2:139–144, 1955.

[Rog58] C. A. Rogers. The packing of equal spheres. *Proc. London Math. Soc. (3)*, 8:609–620, 1958.

[Ush06] Akira Ushijima. A volume formula for generalised hyperbolic tetrahedra. In *Non-Euclidean geometries*, volume 581 of *Math. Appl. (N. Y.)*, pages 249–265. Springer, New York, 2006.

Dept. of Mathematics, College of Charleston, 66 George St., Charleston, SC 29424
E-mail address: `przeworskia@cofc.edu`
URL: `http://przeworskia.people.cofc.edu`