VOLUMES OF HYPERBOLIC 3-MANIFOLDS OF BETTI NUMBER AT LEAST 3

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Abstract. We provide a new, simpler proof of the fact that the smallest volume hyperbolic 3-manifold has betti number at least 3. In the process, we improve the best known lower bound on the volume of such a manifold.

Since it was shown that there is a hyperbolic 3-manifold of smallest volume, there has been great interest in determining this manifold. Conjecturally, it is the Weeks manifold W, obtained by (-5/1,-5/2) Dehn surgery on the Whitehead link. The volume of the Weeks manifold is $0.9427\ldots$. As of now, the best known lower bound on volume for an arbitrary orientable hyperbolic 3-manifold is 0.28, established in [10]. Much better results can be achieved by placing some restrictions on the types of manifolds under consideration [2][6][7][8]. One such result, due to Culler, Hersonsky, and Shalen [4], is that if an orientable hyperbolic 3-manifold has betti number at least 3, then its volume is at least 0.94689 and hence it is not the smallest hyperbolic 3-manifold. We will provide a slight improvement of this result, as well as a much simpler proof.

First, we cite the necessary background results. Rather than repeat the preliminary arguments from [4] we condense these arguments into the following result, which draws upon [3] and [5]:

Theorem 1. *If M is a closed orientable hyperbolic 3-manifold of betti number at least 3 whose shortest geodesic has length at most λ then*

$$Vol(M) \geq V(\lambda) = \frac{\pi\lambda}{e^{\lambda}-1}\left(\frac{e^{2\lambda}+e^{\lambda}+5}{2(\cosh\frac{\lambda}{2})(e^{\lambda}+3)}\right) - \frac{\pi\lambda}{2}$$

Our simplification of [4] will be facillitated by a result of Gabai, Meyerhoff, and Thurston [9]

Theorem 2. [9] *Every closed orientable hyperbolic 3-manifold except Vol3 contains an embedded tube of radius at least* $0.52959\ldots$ *about its*

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shortest geodesic. Unless the shortest geodesic has length greater than $1.0595\ldots$*, there is a tube of radius* $\frac{\log 3}{2}$ *about this geodesic.*

These two results are all that we need.

Theorem 3. *If the first betti number of a closed orientable hyperbolic 3-manifold* M *is at least 3 then the manifold has volume at least* 1.015.

Proof. Since M has betti number at least 3, M is not Vol3. Let l be the length of the shortest geodesic in M. If $l \geq 1.0595$ then by Theorem 2, there is a tube of radius at least 0.52959 about the shortest geodesic. The volume of this tube is then at least $\pi 1.0595 \sinh^2 0.52959 = 1.024\ldots$. If $0.9698 \leq l \leq 1.0595$, then there is a tube of radius at least $\frac{\log 3}{2}$ about the shortest geodesic. The volume of this tube is at least $\pi 0.9698 \sinh^2 \frac{\log 3}{2} = 1.0155\ldots$. If $l \leq 0.9698$, then by Theorem 1, the volume of the maximal tube about the shortest geodesic is at least $V(0.9698) = 1.0158\ldots$. □

A simple corollary of this is:

Corollary 4. *The smallest volume orientable hyperbolic 3-manifold has betti number at most* 2.

Proof. In [1], it is shown that the smallest volume noncompact hyperbolic 3-manifold has volume $1.014\ldots > \mathrm{Vol}(W)$. Hence, the smallest volume orientable hyperbolic 3-manifold must be closed. If it has betti number at least 3, then its volume would be at least $1.015 > \mathrm{Vol}(W)$, a contradiction. □

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