BALLS IN HYPERBOLIC 3-MANIFOLDS

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Abstract. We show that in a closed orientable hyperbolic 3-manifold, any maximal embedded tube of radius r contains a ball of a certain radius. We then use the fact that most closed orientable hyperbolic 3-manifolds contain tubes of radius $\frac{\log 3}{2}$ to provide a universal lower bound on the radius of the ball.

1. Introduction

In this paper we describe a technique for finding an embedded ball which lies within a maximal embedded tube in a closed orientable hyperbolic 3-manifold. This is done by placing a point at a suitable location within the tube and then considering the closest translate of the point by the action along the geodesic at the core of the tube. The main result is a lower bound on the radius of an embedded ball in a closed orientable hyperbolic 3-manifold.

We start with some notation and the relevant prior results. Let $\Gamma \subset PSL_2(\mathbb{C})$ be the fundamental group of a compact orientable hyperbolic 3-manifold M. Choose $\gamma \in \Gamma$ corresponding to a geodesic g in M. Through a large computer search, Gabai, Meyerhoff, and Thurston [GMT03] have shown that there is a tube of radius at least $\frac{\log 3}{2}$ about the shortest geodesic unless M satisfies one of a few exceptional conditions. Suppose there is a maximal tube of radius r about g. Gehring and Martin [GM98] show that this tube has volume at least

$$V(r) = \sqrt{3}\tanh r \cosh 2r \left(\sinh^{-1}\left(\frac{\sinh r}{\cosh 2r}\right)\right)^2.$$

Lifting the tube to $\mathbb{H}^3$, the universal cover of M, we have a maximal tube of radius r about a line. Then γ corresponds to an isometry of $\mathbb{H}^3$ which may be chosen to have this line as its axis. Let the complex length of γ be $l + i\theta$. Then a portion of the tube of length l maps bijectively to the tube in M. Such a portion of a tube has volume $\pi l \sinh^2 r$. Hence we see that $l \geq \frac{V(r)}{\pi \sinh^2 r}$. We also use a result of Meyerhoff [Mey87] concerning radii of tubes.

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Proposition 1.1 ([Mey87]). *If $k = \cosh l - \cos\theta < \sqrt{2} - 1$, then there is a tube of radius R about g where R is given by*

$$\sinh^2 R = \frac{1}{2}\left(\frac{\sqrt{1-2k}}{k} - 1\right).$$

Further, R is a decreasing function of k.

From this, it follows that R will be bigger than r if k is too small. Let $k(r)$ denote the value of k which would yield a tube of radius r. Then $k(r)$ is decreasing in r.

We consider a ball contained within the tube. By requiring that the ball actually touch the boundary of the tube and that it touch some translate of itself under the action of $< \gamma >$, we obtain the largest possible ball which fits inside the tube. It is in this process that we use the estimates provided by [GM98] and [Mey87]. We then produce an expression which determines the radius of the ball given the radius of the tube. This expression is then shown to be increasing in r. As most closed hyperbolic 3-manifolds are known to contain tubes of radius $\frac{\log 3}{2}$, we then have

Theorem 2.12. Any closed orientable hyperbolic 3-manifold contains a ball of radius $0.175\ldots$

There is a comparable result of Friedland [Fri96] which locates embedded balls of radius 0.17 in noncompact hyperbolic 3-manifolds. Higher dimensional results are also developed.

There are earlier results of this nature, such as [Mey87] and [Wat84]. A universal lower bound on the radius of an embedded ball in hyperbolic 3-manifolds provides evidence for the value of the Margulis number (see [Thu78]).

We note that since this article was originally written, various components have been superseded. Gehring and Martin's result [GM98] has been improved. See [Prz03], [GMM01], and [MM03]. Presumably, one could use one of these results in place of [GM98] to improve our result. The basic outline would remain the same, although the computations would have to be redone (and would be considerably more complicated). However, it seems unlikely that such an approach would produce a result stronger than [Prz01], which uses a different approach to (among other things) locate a ball of radius $\sinh^{-1}\frac{1}{4} \approx 0.247$ in any closed orientable hyperbolic 3-manifold.

2. A Ball in a Tube

Proposition 2.1. *A maximal tube of radius r about a geodesic contains a ball of radius $\frac{d}{2}$ where d is given by*

$$k(r) = \frac{\cosh d - \cosh\frac{V(r)}{\pi\sinh^2 r}}{\sinh^2(r - \frac{d}{2})}.$$

Proof. Consider the action of γ on $\mathbb{H}^3$. Let P be a point at a distance of $t < r$ from the axis of γ. Then

$$\cosh d(P, \gamma^n(P)) = \cosh nl + \sinh^2 t(\cosh nl - \cos n\theta).$$

We wish to find the largest ball which is contained in the tube. If the largest ball centered at P has radius less than $r - t$, then we can achieve a larger ball by centering it about a point P' which is farther away from the axis. Also, if the radius of the ball is less than $\frac{1}{2} \min_{n \in \mathbb{Z}^+} d(P, \gamma^n(P))$ then we can achieve a larger ball by choosing it about a point P' which is closer to the axis than P, except when P is already on the axis. If the axis is the optimal location for P, then it must be the case that a ball of radius r embeds in the tube. Since this is much stronger than the result we are trying to prove, we ignore this case. Hence we assume that the radius of the ball is $r - t = \frac{1}{2} \min_{n \in \mathbb{Z}^+} d(P, \gamma^n(P))$.

Let n be the positive integer which minimizes $d(P, \gamma^n(P))$ and let $d_n = d(P, \gamma^n(P)) = 2(r - t)$. With this definition, $t = r - \frac{d_n}{2}$ and

$$\begin{aligned}
\cosh d_n &= \cosh nl + \sinh^2(r - \frac{d_n}{2})(\cosh nl - \cos n\theta) \\
&\geq \cosh \frac{V(r)}{\pi \sinh^2 r} + \sinh^2(r - \frac{d_n}{2})(\cosh nl - \cos n\theta)
\end{aligned}$$

leading to the inequality

$$k_n = \cosh nl - \cos n\theta \leq \frac{\cosh d_n - \cosh \frac{V(r)}{\pi \sinh^2 r}}{\sinh^2(r - \frac{d_n}{2})}$$

From our earlier discussion, it follows that if

$$k_n \leq \frac{\cosh d_n - \cosh \frac{V(r)}{\pi \sinh^2 r}}{\sinh^2(r - \frac{d_n}{2})} < k(r)$$

then we have a tube of radius greater than r, contradicting the maximality of r. Hence, we have that

$$0 < k(r) \leq \frac{\cosh d_n - \cosh \frac{V(r)}{\pi \sinh^2 r}}{\sinh^2(r - \frac{d_n}{2})}.$$

It is easy to see that $\frac{\cosh d_n - \cosh \frac{V(r)}{\pi \sinh^2 r}}{\sinh^2(r - \frac{d_n}{2})}$ is increasing in d_n for $0 < \frac{V(r)}{\pi \sinh^2 r} \leq l \leq d_n < 2r$. Hence there is some minimal value d of d_n for which

$$k(r) = \frac{\cosh d - \cosh \frac{V(r)}{\pi \sinh^2 r}}{\sinh^2(r - \frac{d}{2})}.$$

As $d_n \geq d$, any maximal tube of radius r about a geodesic must contain a ball of radius at least $\frac{d}{2}$. □

Given a particular value for r, it is a simple matter to solve for d. It would seem intuitive that a larger radius tube could fit a larger radius ball, but a larger radius tube may have a shorter length. Thus, we need to verify that d increases as a function of r. First, we develop a more manageable means of determining d.

Proposition 2.2. *$d(r)$ is determined by the equation*

$$0 = a(r)\tanh^2\frac{d}{2} + \tanh\frac{d}{2} + c(r)$$

where $a(r) = \dfrac{(1+\cosh\frac{V(r)}{\pi\sinh^2 r} - \frac{k}{2}\cosh 2r - \frac{k}{2})}{k\sinh 2r}$ *and*
$c(r) = \dfrac{(1-\cosh\frac{V(r)}{\pi\sinh^2 r} - \frac{k}{2}\cosh 2r + \frac{k}{2})}{k\sinh 2r}$.

Proof. We start with the definition of d.

$$k(r) = \frac{\cosh d - \cosh\frac{V(r)}{\pi\sinh^2 r}}{\sinh^2(r-\frac{d}{2})}$$

Cross multiplying and applying half and double angle formulas, we get

$$\begin{aligned}
1+2\sinh^2\frac{d}{2} - \cosh\frac{V(r)}{\pi\sinh^2 r} &= \frac{k}{2}(\cosh(2r-d)-1)\\
&= \frac{k}{2}(\cosh 2r\cosh d - \sinh 2r\sinh d - 1)\\
&= \frac{k}{2}((2\sinh^2\frac{d}{2}+1)\cosh 2r - 2\sinh\frac{d}{2}\cosh\frac{d}{2}\sinh 2r - 1).
\end{aligned}$$

Now, collecting like terms and introducing the equality $1 = \cosh^2\frac{d}{2} - \sinh^2\frac{d}{2}$, we get

$$\begin{aligned}
0 &= (2-k\cosh 2r)\sinh^2\frac{d}{2}\\
&\quad +k\sinh\frac{d}{2}\cosh\frac{d}{2}\sinh 2r + 1 - \cosh\frac{V(r)}{\pi\sinh^2 r} - \frac{k}{2}\cosh 2r + \frac{k}{2}\\
0 &= (2-k\cosh 2r)\sinh^2\frac{d}{2} + k\sinh\frac{d}{2}\cosh\frac{d}{2}\sinh 2r\\
&\quad +(\cosh^2\frac{d}{2} - \sinh^2\frac{d}{2})(1-\cosh\frac{V(r)}{\pi\sinh^2 r} - \frac{k}{2}\cosh 2r + \frac{k}{2})\\
0 &= (1+\cosh\frac{V(r)}{\pi\sinh^2 r} - \frac{k}{2}\cosh 2r - \frac{k}{2})\sinh^2\frac{d}{2} + k\sinh\frac{d}{2}\cosh\frac{d}{2}\sinh 2r\\
&\quad +(1-\cosh\frac{V(r)}{\pi\sinh^2 r} - \frac{k}{2}\cosh 2r + \frac{k}{2})\cosh^2\frac{d}{2}.
\end{aligned}$$

Next, we divide by $k\sinh 2r\cosh^2\frac{d}{2}$.

$$0 = \frac{(1+\cosh\frac{V(r)}{\pi\sinh^2 r} - \frac{k}{2}\cosh 2r - \frac{k}{2})}{k\sinh 2r}\tanh^2\frac{d}{2} + \tanh\frac{d}{2} + \frac{(1-\cosh\frac{V(r)}{\pi\sinh^2 r} - \frac{k}{2}\cosh 2r + \frac{k}{2})}{k\sinh 2r}$$

□

In order to show that d increases with r, we will need some computational lemmas. We start by developing some properties of $k(r)$.

Lemma 2.3. *The following identities hold:*

(1) $k\cosh 2r = \sqrt{1-2k}$
(2) $k\sinh 2r = \sqrt{1-2k-k^2}$
(3) $k\cosh^2 r = \frac{\sqrt{1-2k}+k}{2}$.

Proof. Each follows immediately from the relation

$$\sinh^2 r = \frac{1}{2}\left(\frac{\sqrt{1-2k}}{k} - 1\right).$$

□

We note that if x is defined to satisfy $ax^2+x+c=0$ then $\frac{dx}{dr} = -\frac{a'x^2+c'}{2ax+1}$ and hence we prove

Lemma 2.4. $a(r) > 0$

Proof.

$$a(r) = \frac{1-\frac{1}{2}\sqrt{1-2k} - \frac{k}{2} + \cosh\frac{V(r)}{\pi\sinh^2 r}}{\sqrt{1-2k-k^2}} > 0$$

since $k < \sqrt{2}-1 < \frac{1}{2}$. □

Lemma 2.5. *The function* $\dfrac{1-\frac{k}{2}\cosh 2r}{k\sinh 2r}$ *is decreasing in* r.

Proof.

$$\frac{1-\frac{k}{2}\cosh 2r}{k\sinh 2r} = \frac{1-\frac{1}{2}\sqrt{1-2k}}{\sqrt{1-2k-k^2}}$$

which is increasing as a function of k. Since k is decreasing with r, this expression is also decreasing with r. □

Proposition 2.6. *If* $0.173 < x < 1$ *and* $r \geq \frac{\log 3}{2}$ *then*

$$a'x^2 + c' \leq (1-x^2)\left(\frac{1.06 - \cosh\frac{V(r)}{\pi\sinh^2 r}}{k\sinh 2r}\right)'.$$

Proof.

$$\begin{aligned} a'x^2+c' &= \left(\frac{1-\frac{k}{2}\cosh 2r}{k\sinh 2r}\right)'(x^2+1)+\left(\frac{\cosh\frac{V(r)}{\pi\sinh^2 r}-\frac{k}{2}}{k\sinh 2r}\right)'(x^2-1) \\ &= (1-x^2)\left[\left(\frac{1-\frac{k}{2}\cosh 2r}{k\sinh 2r}\right)'\frac{1+x^2}{1-x^2}-\left(\frac{\cosh\frac{V(r)}{\pi\sinh^2 r}-\frac{k}{2}}{k\sinh 2r}\right)'\right] \end{aligned}$$

As $(1-x^2)\geq 0$, we may disregard it.

Continuing,

$$\begin{aligned} &\left(\frac{1-\frac{k}{2}\cosh 2r}{k\sinh 2r}\right)'\frac{1+x^2}{1-x^2}-\left(\frac{\cosh\frac{V(r)}{\pi\sinh^2 r}-\frac{k}{2}}{k\sinh 2r}\right)' \\ &< \left(\frac{1-\frac{k}{2}\cosh 2r}{k\sinh 2r}\right)'\frac{1+0.173^2}{1-0.173^2}-\left(\frac{\cosh\frac{V(r)}{\pi\sinh^2 r}-\frac{k}{2}}{k\sinh 2r}\right)' \\ &< 1.06\left(\frac{1-\frac{k}{2}\cosh 2r}{k\sinh 2r}\right)'-\left(\frac{\cosh\frac{V(r)}{\pi\sinh^2 r}-\frac{k}{2}}{k\sinh 2r}\right)' \\ &= \left(\frac{1.06-\cosh\frac{V(r)}{\pi\sinh^2 r}}{k\sinh 2r}\right)'+\frac{1}{2}\left(\frac{1-1.06\cosh 2r}{\sinh 2r}\right)'. \end{aligned}$$

It is a matter of elementary calculus to see that the second term is negative when $r\geq\frac{\log 3}{2}$. $\square$

We now focus our efforts on proving that $\dfrac{1.06-\cosh\frac{V(r)}{\pi\sinh^2 r}}{k\sinh 2r}$ is decreasing. We achieve this by breaking it up into successively simpler functions. First, the definition of $V(r)$ is somewhat complicated. We define a similar function

$$\tilde{V}(r)=\sqrt{3}\tanh r\cosh 2r\left(\frac{\sinh r}{\cosh 2r}\right)^2=\sqrt{3}\frac{\tanh r\sinh^2 r}{\cosh 2r}.$$

It is clear that $\tilde{V}(r)>V(r)$. We will require a further result concerning these functions.

Proposition 2.7. *Both* $\frac{\tilde{V}(r)}{\pi\sinh^2 r}$ *and* $\frac{V(r)}{\pi\sinh^2 r}$ *are decreasing for* $r\geq\frac{\log 3}{2}$.

Proof. It is easy to see that $\dfrac{\tilde{V}(r)}{\pi\sinh^2 r}=\dfrac{\sqrt{3}\tanh r}{\pi\cosh 2r}$ achieves its largest value of $0.165\ldots$ when $\cosh 2r=\frac{1+\sqrt{5}}{2}$ and is decreasing for any larger value of r.

We now work with $\frac{V(r)}{\pi\sinh^2 r}$

$$\frac{V(r)}{\pi\sinh^2 r}=\frac{2\sqrt{3}}{\pi}\coth 2r\left(\sinh^{-1}(\frac{\sinh r}{\cosh 2r})\right)^2.$$

This is obviously decreasing whenever $\dfrac{\sinh r}{\cosh 2r}$ is. Thus, we suppose that $\dfrac{\sinh r}{\cosh 2r}$ is increasing. Then

$$\frac{V(r)}{\pi \sinh^2 r} = \frac{V(r)}{\tilde{V}(r)} \frac{\tilde{V}(r)}{\pi \sinh^2 r} = \left(\frac{\sinh^{-1}(\frac{\sinh r}{\cosh 2r})}{\frac{\sinh r}{\cosh 2r}} \right)^2 \frac{\tilde{V}(r)}{\pi \sinh^2 r}$$

is decreasing. $\square$

We now, in effect, replace V with $\tilde{V}$.

$$(1) \quad \frac{1.06 - \cosh \frac{V(r)}{\pi \sinh^2 r}}{k \sinh 2r} = \frac{1.06 - \cosh \frac{\tilde{V}(r)}{\pi \sinh^2 r}}{k \sinh 2r} + \frac{\cosh \frac{\tilde{V}(r)}{\pi \sinh^2 r} - \cosh \frac{V(r)}{\pi \sinh^2 r}}{k \sinh 2r}$$

Consider the first term of the right side of (1).

$$\frac{1.06 - \cosh \frac{\tilde{V}(r)}{\pi \sinh^2 r}}{k \sinh 2r} = \left(\frac{1}{2k \cosh^2 r} \right) \left[(\coth r)(1.06 - \cosh \frac{\tilde{V}(r)}{\pi \sinh^2 r}) \right]$$

From Lemma 2.3, we see that

$$\frac{1}{2k \cosh^2 r} = \frac{1}{\sqrt{1 - 2k} + k}$$

which is increasing in k so decreasing in r. The other part of this expression requires more work.

Lemma 2.8. *The function* $(\coth r)(1.06 - \cosh \frac{\tilde{V}(r)}{\pi \sinh^2 r})$ *is decreasing when* $r \geq \frac{\log 3}{2}$.

Proof.

$$\begin{aligned}
\frac{d}{dr} \left[(\coth r)(1.06 - \cosh \frac{\tilde{V}(r)}{\pi \sinh^2 r}) \right] &= -\operatorname{csch}^2 r \left(1.06 - \cosh \left(\frac{\sqrt{3}}{\pi} \frac{\tanh r}{\cosh 2r} \right) \right) \\
&\quad - \frac{\sqrt{3}}{\pi} \coth r \sinh \left(\frac{\sqrt{3}}{\pi} \frac{\tanh r}{\cosh 2r} \right) \left(\frac{\cosh 2r \operatorname{sech}^2 r - 2 \tanh r \sinh 2r}{\cosh^2 2r} \right) \\
&= \operatorname{csch}^2 r \left[-1.06 + \cosh \left(\frac{\sqrt{3}}{\pi} \frac{\tanh r}{\cosh 2r} \right) \right. \\
&\quad \left. - \frac{\sqrt{3}}{\pi} \tanh r \sinh \left(\frac{\sqrt{3}}{\pi} \frac{\tanh r}{\cosh 2r} \right) \left(\frac{\cosh 2r - \sinh^2 2r}{\cosh^2 2r} \right) \right]
\end{aligned}$$

Since $\dfrac{\tilde{V}(r)}{\pi \sinh^2 r} = \dfrac{\sqrt{3} \tanh r}{\pi \cosh 2r}$ is decreasing for $r \geq \frac{\log 3}{2}$ we have $\cosh \left(\dfrac{\sqrt{3} \tanh r}{\pi \cosh 2r} \right) < 1.014$ and $\sinh \left(\dfrac{\sqrt{3} \tanh r}{\pi \cosh 2r} \right) < 1.005 \dfrac{\sqrt{3} \tanh r}{\pi \cosh 2r}$.

Since $\cosh 2r - \sinh^2 2r < 0$ for $r \geq \frac{\log 3}{2}$ we have

$$\begin{aligned}
&\frac{d}{dr}\left[\coth r(1.06 - \cosh\frac{\tilde{V}(r)}{\pi\sinh^2 r})\right]\\
&< \operatorname{csch}^2 r\left(-1.06 + 1.014 - 1.005\cdot\frac{3}{\pi^2}\tanh^2 r\frac{\cosh 2r - \sinh^2 2r}{\cosh^3 2r}\right)\\
&< \operatorname{csch}^2 r\left(-1.06 + 1.014 + 1.005\cdot\frac{3}{\pi^2}\frac{\tanh^2 r\sinh^2 2r}{\cosh^3 2r}\right)\\
&= \operatorname{csch}^2 r\left(-1.06 + 1.014 + 1.005\cdot\frac{3}{\pi^2}\frac{\cosh 2r - 1}{\cosh 2r + 1}\cdot\frac{\cosh^2 2r - 1}{\cosh^3 2r}\right)\\
&= \operatorname{csch}^2 r\left(-1.06 + 1.014 + 1.005\cdot\frac{3}{\pi^2}\frac{(\cosh 2r - 1)^2}{\cosh^3 2r}\right)\\
&\leq \operatorname{csch}^2 r\left(-1.06 + 1.014 + 1.005\cdot\frac{3}{\pi^2}\cdot\frac{4}{27}\right)\\
&< 0
\end{aligned}$$

as $\frac{(y-1)^2}{y^3} \leq \frac{4}{27}$ when $y \geq 1$. □

We now turn our attention to the second term of (1).

Proposition 2.9. *The function*

$$\frac{\cosh\frac{\tilde{V}(r)}{\pi\sinh^2 r} - \cosh\frac{V(r)}{\pi\sinh^2 r}}{k\sinh 2r}$$

is decreasing for $r \geq \frac{\log 3}{2}$.

Proof.

$$\frac{\cosh\frac{\tilde{V}(r)}{\pi\sinh^2 r} - \cosh\frac{V(r)}{\pi\sinh^2 r}}{k\sinh 2r} = \frac{\cosh\frac{\tilde{V}(r)}{\pi\sinh^2 r} - \cosh\frac{V(r)}{\pi\sinh^2 r}}{\frac{\tilde{V}(r)}{\pi\sinh^2 r} - \frac{V(r)}{\pi\sinh^2 r}}\cdot\frac{\tilde{V}(r) - V(r)}{\pi k\sinh 2r\sinh^2 r}$$

It is easy to see that $\dfrac{\cosh u_1 - \cosh u_2}{u_1 - u_2}$ is decreasing if u_1 and u_2 are. Thus, we're left with only

$$\frac{\tilde{V}(r) - V(r)}{\pi k\sinh 2r\sinh^2 r}.$$

If $\dfrac{\sinh r}{\cosh 2r}$ is decreasing, then we have

$$\frac{\tilde{V}(r) - V(r)}{\pi k\sinh 2r\sinh^2 r} = \frac{\tilde{V}(r)}{\pi\sinh^2 r}\cdot\frac{1}{k\sinh 2r}\cdot\left(1 - \left(\frac{\sinh^{-1}(\frac{\sinh r}{\cosh 2r})}{\frac{\sinh r}{\cosh 2r}}\right)^2\right)$$

which is decreasing. Thus, we assume $\dfrac{\sinh r}{\cosh 2r}$ is increasing.

$$\frac{\tilde{V}(r) - V(r)}{\pi k \sinh 2r \sinh^2 r} = \frac{\tilde{V}(r)}{\pi \sinh^2 r} \cdot \left(\frac{\left(\frac{\sinh r}{\cosh 2r}\right)^2}{k \sinh 2r} \right) \left(\frac{\left(\frac{\sinh r}{\cosh 2r}\right)^2 - \left(\sinh^{-1} \frac{\sinh r}{\cosh 2r}\right)^2}{\left(\frac{\sinh r}{\cosh 2r}\right)^4} \right)$$

The first term $\dfrac{\tilde{V}(r)}{\pi \sinh^2 r}$ is known to be decreasing. The second term is

$$\frac{\left(\frac{\sinh r}{\cosh 2r}\right)^2}{k \sinh 2r} = \frac{\left(\frac{\sinh r}{\cosh 2r}\right)^2 \coth 2r}{k \sinh 2r \coth 2r} = \frac{1}{2\sqrt{3}} \cdot \frac{\tilde{V}(r)}{\sinh^2 r} \cdot \frac{1}{k \cosh 2r}$$

which is decreasing. Finally the last term is of the form $\dfrac{u^2 - (\sinh^{-1} u)^2}{u^4}$ which is decreasing in u and hence in r. □

Proposition 2.10. *The function $d(r)$ is increasing for $r \geq \frac{\log 3}{2}$.*

Proof. To determine $d(r)$ we must solve an expression of the form $a(r)x^2 + x + c(r) = 0$ where $x = \tanh \frac{d}{2}$. To see that x is increasing in r, we differentiate with respect to r yielding $\dfrac{dx}{dr} = -\dfrac{a'x^2 + c'}{2ax + 1}$. When $r = \frac{\log 3}{2}$, $x = 0.173896\ldots$. If we can show that $\dfrac{dx}{dr} > 0$ when $x > 0.173$ then it will follow that $\dfrac{dx}{dr} > 0$ if $r \geq \frac{\log 3}{2}$. Hence we may assume $x > 0.173$. As a and x are known to be positive and

$$a'x^2 + c' \leq (1 - x^2) \left(\frac{1.06 - \cosh \frac{V(r)}{\pi \sinh^2 r}}{k \sinh 2r} \right)' \leq 0$$

it follows that x, and hence d, is an increasing function of r. □

Corollary 2.11. *Any tube of radius $r \geq \frac{\log 3}{2}$ about a geodesic contains a ball of radius at least $\frac{1}{2} d(\frac{\log 3}{2}) = 0.175\ldots$*

Theorem 2.12. *Any closed orientable hyperbolic 3-manifold contains a ball of radius $0.175\ldots$*

Proof. In [GMT03] it is shown that the shortest geodesic in the manifold has a tube of radius at least 0.52955 or the manifold is Vol3. Further, it is shown that if the tube radius is less than $\frac{\log 3}{2}$ then the geodesic has length at least 1.0595. We have established the result in the case where the tube radius is at least $\frac{\log 3}{2}$. Vol3 is known to contain a ball of radius $0.527\ldots$. A tube of radius 0.52955 and length at least $1.0595 \geq 2 \cdot 0.52955$ contains a ball of radius 0.52955 □

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